Dimensional Analysis

- Units and dimensions
- Dimensional Analysis
- Buckingham Pi theory
- Benefits of non-dimensionalization
- examples

Learning Objectives – Dimensional Analysis

By the end of this section, you should be able to:

- determine whether equations are dimensionally homogeneous.
- check the units of a solution they have reached and verify that they are correct.
- use dimensional analysis to determine the appropriate units for an unknown quantity in an equation.
- apply dimensional analysis to predict formulas which connect particular variables in given circumstances
Dimensional Analysis: rationale

- Physical laws must be independent of arbitrarily chosen units of measure. (In other words, Newton's second law, $F = ma$, is true whether we choose to measure mass in kilograms, acceleration in $m/s^2$, and force in newtons, or whether we measure mass in slugs, acceleration in $ft/s^2$, and force in pounds.)

- Consideration of dimensions can guide us in the discovery of physical laws.

- (Check your units! All natural/physical relations must be dimensionally correct.)

Units and Dimensions

While units are arbitrarily chosen, dimensions represent fundamental quantities.

<table>
<thead>
<tr>
<th>Dimension</th>
<th>Unit</th>
<th>Symbol</th>
</tr>
</thead>
<tbody>
<tr>
<td>Length</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Time</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Mass</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Temperature</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Electrical charge</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Units

The ability to use units correctly is essential for scientists and engineers. Mistakes can be expensive: example: What caused NASA to lose the Mars Climate Orbiter probe in 1999?

(A) It was hit by a meteorite
(B) Aliens
(C) A mix up over units
(D) Metal fatigue in a booster

You should always state the units that you are using.

The dimensions of a physical quantity $x$ are denoted by $[x]$. Derive the dimensions of the following physical quantities:

1. velocity = distance/time 
   $[v] = LT^{-1}$
2. acceleration = velocity/time
3. force = mass $\times$ acceleration
4. power = rate of working = work/time = (force $\times$ distance)/time
5. dynamic viscosity = shear stress/velocity gradient = force/area/velocity gradient
Units

Derive the dimensions of the following physical quantities:

6. electric current =

7. electric field =

8. electric charge =

9. electric power =

10. energy =

11. mass density =

Derived Units

Some physical quantities are so widely used that their units have special names. They are called derived units because they are defined in terms of the base SI units. The SI unit of force is the Newton (N). Let us see how it is constructed in terms of base units. Force is defined by:

\[ \text{Force} = \text{mass} \times \text{acceleration} \]
\[ \Rightarrow \text{Units of force} = \text{kg} \times \text{m s}^{-2} = \text{kg m s}^{-2} \]

One Newton is short for one kilogram meter per second squared, i.e., it is the force needed to accelerate one kilogram by one meter per second squared.

example: The SI unit of energy is the Joule (J).

From the expression for kinetic energy \( E = \frac{1}{2} mv^2 \), where \( m \) is the mass and \( v \) is the speed of the object, express the Joule in terms of the base SI units.

(A) kg m^2 s^{-2}  \quad (B) kg^2 m^{-2} s^{-2}  \quad (C) kg m s  \quad (D) kg^{-1} m s^{-2}
Dimensionless quantities

Some quantities are dimensionless. These are pure numbers which would be the same regardless of the units used.

Examples:
- Trigonometric functions
- Logarithms
- Exponentials
- Numbers ($\pi$)

Functions of dimensionless variables are dimensionless.
Dimensionless functions must have dimensionless arguments.

Checking equations

Any terms which are added together or subtracted must have the same dimensions.
The dimensions of both sides of an equation must agree.

Such a scientific equation is dimensionally correct.
Although dimensional analysis is a way of checking that equations might be true, it does not prove that they are definitely correct.
Dimensional analysis would suggest that Einstein’s equation could be either $E = mc^2$ or $E = \frac{1}{2} mc^2$. However, dimensional analysis shows that $E = mc^3$ is incorrect.
Examples

1. Hooke’s law states that the force F in a spring extended by a length x is given by \( F = -kx \). From Newton’s second law \( F = ma \), where m is the mass and a is the acceleration. Calculate the dimension of the spring constant k.

(A) \( MT^{-2} \)  
(B) \( M T^2 \)  
(C) \( M L^{-2} T^{-2} \)  
(D) \( M L^2 T^2 \)

2. The expressions for kinetic energy \( E = \frac{1}{2} mv^2 \) and potential energy \( E = mgh \) look different but they both describe energy. Calculate the dimensions of both.

\( \frac{1}{2} mv^2 \)  
\( mgh \)

3. The Boltzmann distribution in thermodynamics involves the factor \( \exp(-E/(kT)) \) where E represents energy, T is the temperature and k is Boltzmann’s constant. Find the dimensions of k.

Gulliver’s Travels: Dimensional Analysis
Dimensional Analysis

Dimensional Analysis uses information from the fundamental dimensions that describe a process or problem and use that information to gain insight into the problem. It is based on dimensional consistency or dimensional homogeneity – the principle that all additive or equated terms in an equation must have the same net dimensions. Dimensional analysis of a model can indicate the important dimensionless groups of variables, reduce the model to a minimal form, and make it easy to assess asymptotic behavior of the system. In a way, dimensional analysis of a describing equation is the source of the dimensionless groups we look at to understand a problem.
Physical Parameters

All physical variables have units (dimensions). Dimensional analysis relies on the ideas of dimensional consistency:

• in any equation, all terms must have the same dimensions or they cannot be added.
• both sides of an equation must have the same dimensions.
• arguments of many functions (log, sin, exp, etc.) and all exponents must be dimensionless.

Dimensional Analysis Theorem

Dimensional Homogeneity Theorem: Any physical quantity is dimensionally a power law monomial - [Any Physical Quantity] = M^a L^b T^c Q^d

It is almost always useful to carry out a dimensional analysis:
  Fundamentally wrong forms of theories can be avoided
  The amount of experimental validation can be reduced
  The correct similarities between models and full-scale can be identified.
Dimensional Analysis

Any approach to dimensional analysis will
1. establish a set of variables needed to describe a system
   • from a full mathematical model
   • by experience from known relationships and knowledge of the system
2. determine the dimensionless groups which describe the system
   • by transforming the model to dimensionless form
   • by algebraic combination of variables
3. verify the results by experiment

If you start with a full model, the dimensionless form can be used to study limiting behavior of the system -- what happens when one group becomes much larger or much smaller than other -- and the dimensionless groups can be interpreted in terms of fundamental behaviors.

If you start from a list of variables, trying to correlate their relationship, algebraic combination of the variables into dimensionless groups reduces the number of parameters. Then, when you try to fit the data, you need fewer experiments. Be aware, though, that dimensional analysis does not specify the form of the correlating equation.

Procedure

1. Determine the relevant quantities from physical considerations. This may mean identifying the equation(s) that determine how the system behaves (F = ma).
2. Determine the fundamental units ([M], [L], [T]) of each quantity.
3. Postulate an equation relating the quantities, with unknown exponents (a, b, ...)
4. Substitute units.
5. Equate exponents of [M], [L], [T] and solve the resulting equations simultaneously.
6. Check your results by plugging back into the equations.
Example – object falling under gravity

Basic physics of objects falling under gravity. Develop a formula that tells us the time it takes an object to fall a given distance $s$.

1. Determine the relevant quantities from physical considerations.
   - We think that the time will depend on $s$ and also on the acceleration due to gravity $g$.
2. Determine the fundamental units ([M], [L], [T]) of each quantity.
3. Postulate an equation relating the quantities, with unknown exponents $(a, b, ...)$

4. Substitute units.

5. Equate exponents of [M], [L], [T] and solve the resulting equations simultaneously.

6. Check your results by plugging back into the equations.

We can now design an experiment to determine $C$.

This is a very powerful principle and can help us in finding laws or equations and developing theories.

Example – pendulum

Angular frequency of small oscillations of a point pendulum of length $l$ and mass $m$. 
Airfoil design

An airfoil is the shape of a wing as seen in cross-section. An airfoil shaped body moved through a fluid produces a force perpendicular to the motion called lift.

Example – airfoil design

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Symbol</th>
<th>Units</th>
</tr>
</thead>
<tbody>
<tr>
<td>Lift per span</td>
<td>$L'$</td>
<td>M T^{-2}</td>
</tr>
<tr>
<td>Angle of attack</td>
<td>$\alpha$</td>
<td>-</td>
</tr>
<tr>
<td>Freestream velocity</td>
<td>$V_\infty$</td>
<td>L T^{-1}</td>
</tr>
<tr>
<td>Freestream density</td>
<td>$\rho_\infty$</td>
<td>M L^{-3}</td>
</tr>
<tr>
<td>Freestream viscosity</td>
<td>$\mu_\infty$</td>
<td>M L^{-1} T^{-1}</td>
</tr>
<tr>
<td>Freestream speed of sound</td>
<td>$a_\infty$</td>
<td>L T^{-1}</td>
</tr>
<tr>
<td>Size of body</td>
<td>$C$</td>
<td>L</td>
</tr>
</tbody>
</table>

$L' = f (\alpha, V_\infty, \rho_\infty, \mu_\infty, a_\infty, C)$
Buckingham Pi Theory

The Buckingham Pi Theorem states that a functional statement can be rescaled into an equivalent dimensionless statement

$$\Pi_1 = f(\Pi_2, \Pi_3, \ldots \Pi_{N-K})$$

having only $N - K$ dimensionless parameters. These are called Pi products, since they are suitable products of dimensional parameters.

$$\Pi_1 = \frac{L'}{\frac{1}{2}\rho_c V^2_c} = c_l$$ lift coefficient

$$\Pi_2 = \alpha$$ angle of attack

$$\Pi_3 = \frac{\rho_c V_c c}{\mu} = Re$$ Reynolds number

$$\Pi_4 = \frac{V_c}{\mu} = M_c$$ Mach number

Benefits of non-dimensionalization

The dimensionless form then becomes

$$c_l = f(\alpha, Re, M_c)$$

The original 6 dimensional parameters which influence $L'$ has been reduced to only 3 dimensionless parameters which influence $c_l$.

Consider an experiment where the effect of all parameters is to be determined. In this experiment we give each parameter 5 distinct values in order to ascertain its effect on the lift. If we work with the original 6 dimensional parameters, then the number of possible parameter combinations and experimental runs required is:

$$5^6 = 15625$$

If we work with the 3 dimensionless parameters, the number of parameter combinations and experimental runs is only:

$$5^3 = 125$$
Airfoil flow

Consider two airfoils which have the same shape and angle of attack, but have different sizes and are operating in two different fluids.

<table>
<thead>
<tr>
<th>Airfoil 1 (sea level)</th>
<th>Airfoil 2 (cryogenic tunnel)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha_1 = 5^\circ$</td>
<td>$\alpha_2 = 5^\circ$</td>
</tr>
<tr>
<td>$V_1 = 210$ m/s</td>
<td>$V_2 = 140$ m/s</td>
</tr>
<tr>
<td>$\rho_1 = 1.2$ kg/m$^3$</td>
<td>$\rho_2 = 3.0$ kg/m$^3$</td>
</tr>
<tr>
<td>$\mu_1 = 1.8 \times 10^{-5}$ kg/ms</td>
<td>$\mu_2 = 1.5 \times 10^{-5}$ kg/ms</td>
</tr>
<tr>
<td>$a_1 = 300$ m/s</td>
<td>$a_2 = 200$ m/s</td>
</tr>
<tr>
<td>$c_1 = 1.0$ m</td>
<td>$c_2 = 0.5$ m</td>
</tr>
</tbody>
</table>

Calculate the Pi products $\alpha$, $Re$, $M$.

Similarity

*Geometrical similarity*: same shape.
The parameters ($\Pi_1$, $\Pi_2$, ... $\Pi_{N+k}$) are known as *similarity parameters*.

*Dynamic similarity*: the same values for dimensionless groups involving distance, time, inertia (mass)

*Kinematic similarity*: the same values for dimensionless groups involving distance and time

Similar systems have essentially the same physics. Similarity is the fundamental principle underlying all testing with models.
All dimensionless groups must be kept the same in the model and prototype to achieve perfect dynamic similarity.
G. I. Taylor’s 1947 Analysis

The first explosion of an atomic bomb: Trinity test in New Mexico, 1945. The power released by the explosion was a secret.

1 kiloton TNT = 4.184 TJ = 4.18 x 10^{12} J
yield = amount of energy discharged when nuclear weapon is detonated

Energy in a nuclear explosion

In a nuclear explosion there is a nearly instantaneous release of energy $E$ in a small region of space. This produces a spherical shock wave, with the pressure inside the shock wave thousands of times greater than the initial air pressure.

How does the radius $R$ of this shock wave grow with time $t$?

The relevant variables are:

<table>
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<tr>
<th>Parameter</th>
<th>Symbol</th>
</tr>
</thead>
<tbody>
<tr>
<td>radius of wavefront</td>
<td>$R$</td>
</tr>
<tr>
<td>time</td>
<td>$t$</td>
</tr>
<tr>
<td>ambient density</td>
<td>$\rho_0$</td>
</tr>
<tr>
<td>energy released</td>
<td>$E$</td>
</tr>
</tbody>
</table>
Nuclear explosion shock wave

blast radius vs time

Blast Radius vs Time

\log R = 0.4058 \log t + 1.5593