Discrete Structures & Algorithms

# Propositional Logic

EECE 320 // UBC

# Review of last lecture

- Pancake sorting
  - A problem with many applications
  - Bracketing (bounding a function)
  - Proving bounds for pancake sorting
  - You can make money solving such problems (Bill Gates!)
  - Illustrated many concepts that we will learn in this course
    - Proofs
    - Sets
    - Counting
    - Induction
    - Performance of algorithms

# Propositional logic

- Logic of compound statements built from simpler statements using Boolean connectives.
- Building block for mathematics and computing.
- Direct applications
  - Design of digital circuits
  - Expressing conditions in programs
  - Database queries

I find the question, “Why are we here?” typically human. I’d suggest “Are we here?” would be the more logical choice. (Leonard Nimoy)

Against logic there is no armor like ignorance. (Laurence J. Peter)

Logic is like the sword – those who live by it shall perish by it. (Samuel Butler)

# A sandwich is better than God

- Nothing is better than God.
- A sandwich is better than nothing.
- Thus, a sandwich is better than God.

# What is propositional logic?

- The simplest form of mathematical logic.
- Develops a symbolic language to treat compound and complex propositions and their logical relationship in an abstract manner.

- And, before we get ahead of ourselves... what is a proposition?
  - A declarative statement that is either true or false (but not both).

# Examples of propositions

- September 6 2007 is a Thursday.
- September 6 2007 is a Friday.
- 3+2 equals 7.
- There is no gravity.

The following are **not** propositions.

- Do your homework.
- What is the time?
- 3+4.

# Propositions and their negations

Suppose $p$ is a proposition, then the negation of $p$ is written as $\neg p$.

or sometimes as $\bar{p}$

The negation of proposition $p$ implies that "It is not the case that $p$."

Examples

$p$: It is raining. $\neg p$: It is not raining.

$p$: $3+2=5$. $\neg p$: $3+2\neq5$.

Notice that $\neg p$ is a proposition too.

# Conjunctions and disjunctions

- Elaborate ways of saying "and" and "or"
- Consider two propositions, $p$ and $q$
  - Conjunction ("and"): $p \wedge q$
    - It is a bright and windy day.
    - The day has to be both bright and windy.
  - Disjunction ("or"): $p \vee q$
    - To ride the bus you must have a ticket or hold a pass.
    - One of the two conditions ("have a ticket" or "hold a pass") suffices. (Though both could be true.)

# “Exclusive or”

- In day-to-day speech, sometimes we use “or” as an “exclusive or”.
- “I will take a taxi or a bus from the airport.”
  - Only one of “taxi” or “bus” is implied.
  - To be precise, one would need to say “I will take either a taxi or a bus from the airport.”
- “Exclusive or”/XOR is denoted by the symbol $\oplus$.

# Truth tables

- Truth tables can be used to evaluate statements. A simple proposition can either be "true" or "false".

| Negation | |
|---|---|
| $p$ | $\neg p$ |
| T | F |
| F | T |

| Conjunction ("and") | | |
|---|---|---|
| $p$ | $q$ | $p \wedge q$ |
| F | F | F |
| F | T | F |
| T | F | F |
| T | T | T |

| Disjunction ("or") | | |
|---|---|---|
| $p$ | $q$ | $p \vee q$ |
| F | F | F |
| F | T | T |
| T | F | T |
| T | T | T |

Notice that the conjunction and disjunction of two propositions are also propositions (and, along with negation, are called compound propositions).

# Logical equivalence

Two statements are logically equivalent if and only if they have identical truth tables.

The simplest example is $\neg(\neg p) \equiv p$.

# Implication

- Another important logical construct is **implication**, which is a **conditional statement**.
- This is akin to saying "If … then …"
- When proposition $p$ holds then $q$ holds.
  - This is expressed as $p \rightarrow q$.
  - Example: If I am on campus, I study a lot.
    - $p$: I am on campus.
    - $q$: I study a lot.

# Truth table for implication

- $p \rightarrow q$: If $p$, then $q$.
- If $p$ is true, $q$ must be true for the implication to hold.
- $p$ is the assumption/premise/antecedent.
- $q$ is the conclusion/consequent.

| Implication | | |
|---|---|---|
| $p$ | $q$ | $p \rightarrow q$ |
| F | F | T |
| F | T | T |
| T | F | F |
| T | T | T |

# An equivalent for implication

**Is there an expression that is equivalent to $p \rightarrow q$ but uses only the operators $\neg, \wedge, \vee$?**

**Consider the proposition $\neg p \vee q$**

| Implication | | |
|---|---|---|
| $p$ | $q$ | $p \rightarrow q$ |
| F | F | T |
| F | T | T |
| T | F | F |
| T | T | T |

| $p$ | $q$ | $\neg p$ | $\neg p \vee q$ |
|---|---|---|---|
| F | F | T | T |
| F | T | T | T |
| T | F | F | F |
| T | T | F | T |

# Proving other equivalences

- Easy to use truth tables and show logical equivalences.
- Example: distributivity
  - $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$
  - Do this as an exercise.
  - You would have seen these forms in earlier courses on digital logic design.

# Logical equivalences

- The basic laws
  - Identity
  - Domination
  - Idempotence
  - Negation and double negation
  - Commutation
  - Association
  - Distribution
  - Absorption
  - De Morgan’s laws

Propositions that are logically equivalent. You will need to know them, although we will not elaborate on them in lecture.

In the text (Rosen): Chapter 1, Section 2.

# Variations on a proposition

- Given a **proposition** $p \rightarrow q$, there are other propositions that can be stated.
  - Example: If a function is not continuous, it is not differentiable.

> Of the three (contrapositive, converse, inverse), which is not like the other two?

- **Contrapositive:** $\neg q \rightarrow \neg p$
  - Example: If a function is differentiable, then it is continuous.
- **Converse:** $q \rightarrow p$

> Hint: One is a logical equivalent of the original proposition.

  - Example: If a function is not differentiable, then it is not continuous.
- **Inverse:** $\neg p \rightarrow \neg q$

> The "contrapositive" is equivalent to the original proposition.

  - Example: If a function is continuous, then it is differentiable.

# Tautology & Contradiction

- A proposition that is always true is called a **tautology**.
  - Example: $p \lor \neg p$
- A proposition that is always false is called a **contradiction**.
  - Example: $p \land \neg p$

# Bi-conditionals ("if and only if")

- If $p$ and $q$ are two propositions, then $p \leftrightarrow q$ is a bi-conditional proposition.
  - $p$ **if and only if** $q$. ($p$ *iff* $q$)
  - $p$ is necessary and sufficient for $q$.
  - If $p$ then $q$, and conversely.
- **Example: The Thunderbirds win if and only if it is raining.**
- $p \leftrightarrow q$ is the same as $(p \rightarrow q) \wedge (q \rightarrow p)$.
  - Are there other other equivalent statements?

# Necessary and sufficient

- If $p \rightarrow q$, then $p$ is a sufficient condition for $q$.
  - $q$ can be true even when $p$ is not, therefore $p$ is not necessary for $q$ to be true.
- If $p \leftrightarrow q$, then $p$ is a necessary and sufficient condition for $q$.
  - $q$ is true only when $p$ is, and $q$ is always true when $p$ is.

# For arguments' sake

- What is an argument?
  - It is a sequence of statements the ends with a conclusion.
  - Not the common language usage of a debate or dispute.

**Structure of an argument**

Statement 1 ($p_1$)
Statement 2 ($p_2$)
...
Statement n ($p_n$)

premises
or
antecedents

$\therefore$ Statement n+1 (conclusion)

# For arguments' sake

- Premises
  - "If you have a current password, you can log onto the computer network."
  - "You have a current password."
- Conclusion
  - "Therefore, you can log onto the computer network."

# Representing an argument

**An argument is sometimes written as follows:**

$$\begin{array}{l} p_1 \\ p_2 \\ \vdots \\ p_n \\ \hline \therefore q \end{array}$$

premises

conclusion

# Valid arguments

- An argument is **valid** if and only if it is **impossible** for all the **premises to be true** and the **conclusion to be false**.
- How do we show that an argument is valid?
  - We can use a truth table, or
  - We can show that $(p_1 \wedge p_2 \wedge \ldots \wedge p_n \rightarrow p_{n+1})$ is a tautology using some rules of inference.

# Why use rules of inference?

- Constructing a truth table is time consuming!
- If we have $n$ propositions, what is the size of the truth table? $2^n$, which means that the table doubles in size with every proposition.

| Implication | | |
|---|---|---|
| $p$ | $q$ | $p \rightarrow q$ |
| F | F | T |
| F | T | T |
| T | F | F |
| T | T | T |

Two propositions are involved in an implication, therefore the truth table has $2^2 = 4$ rows.

# Rules of inference

$$
\begin{array}{l}
p \rightarrow q \\
p \\
\hline
\therefore q
\end{array}
$$

**modus ponens**

**"method of affirming"**

# Rules of inference

$$\begin{array}{l} p \longrightarrow q \\ \overline{q} \\ \hline \therefore \overline{p} \end{array}$$

**modus tollens**

**“method of denying”**

# Rules of inference

More rules of inference listed in the text. Try proving them as an exercise. Chapter 1, Section 5 (Rosen).

$$\frac{p}{\therefore p \vee q}$$

**generalization**

$$\frac{p \wedge q}{\therefore p}$$

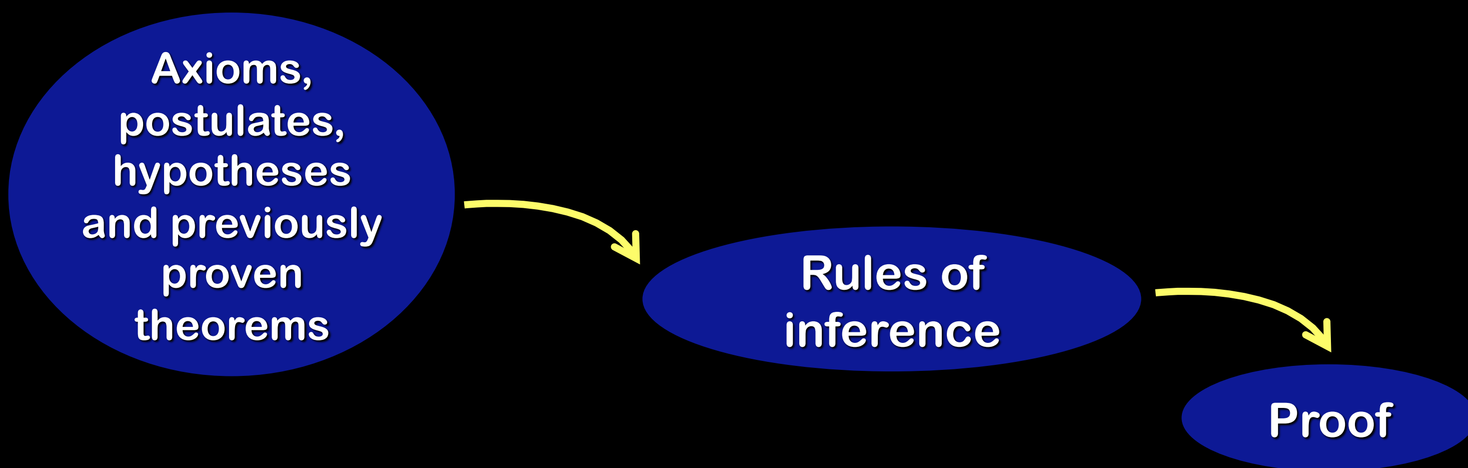
**specialization**

# Proofs

- If he is Mark Spitz then he is a great swimmer.
- How do we prove that he is a great swimmer?
  - If we identify him to be Mark Spitz, then this statement is true: “He is Mark Spitz.”
  - Then this is true: “He is a great swimmer.”
    - by modus ponens

# Proofs and theorems

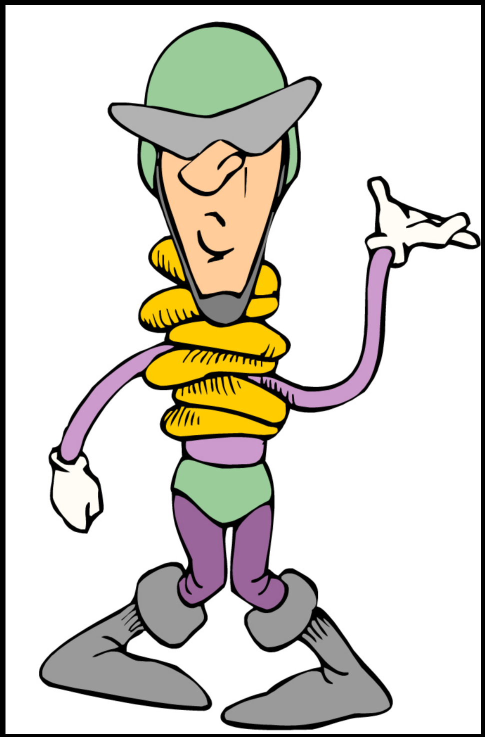
A **theorem** is a statement that can be shown to be true. A **proof** is the means of doing so.



# Pancake numbers

- How did we prove the bounds on $P_n$?
- $n \leq P_n \leq 2n - 3$
- What are the propositions involved?

# Wrap up



- Propositional logic
  - Or propositional calculus
  - Truth tables
  - Logical equivalence
  - Basic laws
- Rules of inference
  - Arguments
  - Premises and conclusions
- Proofs