Discrete Structures & Algorithms

Predicate Logic & Proofs

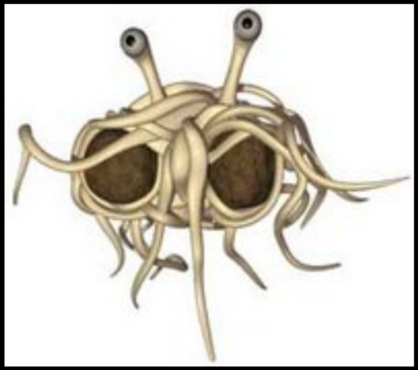
EECE 320 // UBC

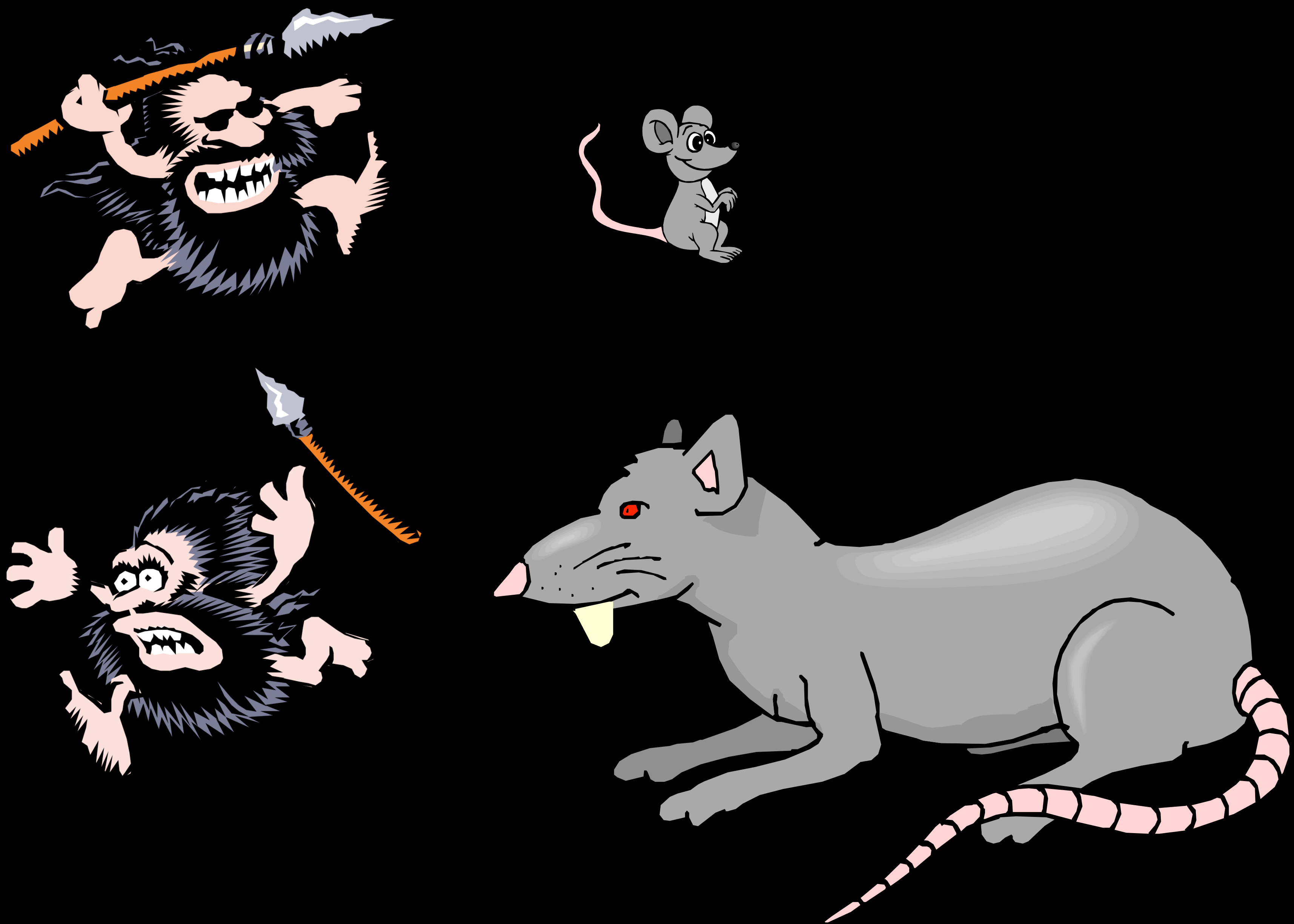
Bits of wisdom

What did our brains evolve to do?

What were our brains
"intelligently designed" to do?

What kind of meat did the Flying
Spaghetti Monster put in our heads?





Not for mathematics

- Our brains did not evolve to do mathematics? (Or computer engineering, for that matter.)
- Over the past 30,000 years or so, our brains have stayed (essentially) the same.

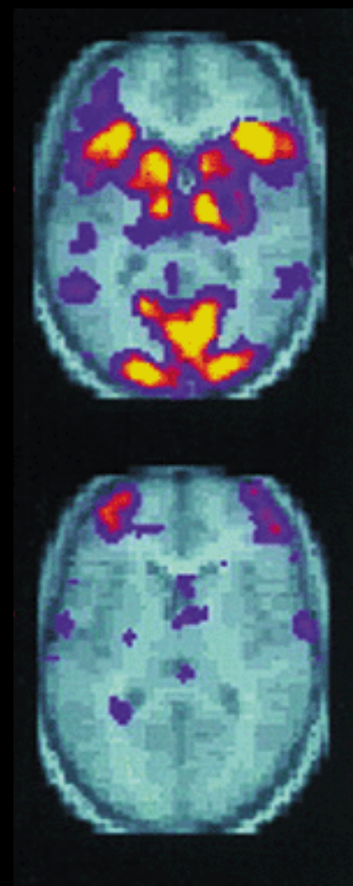
Our brains can perform only simple, concrete tasks.

And that's how math should be approached!

Substitute concrete values for the variables: x=0, x=100, …

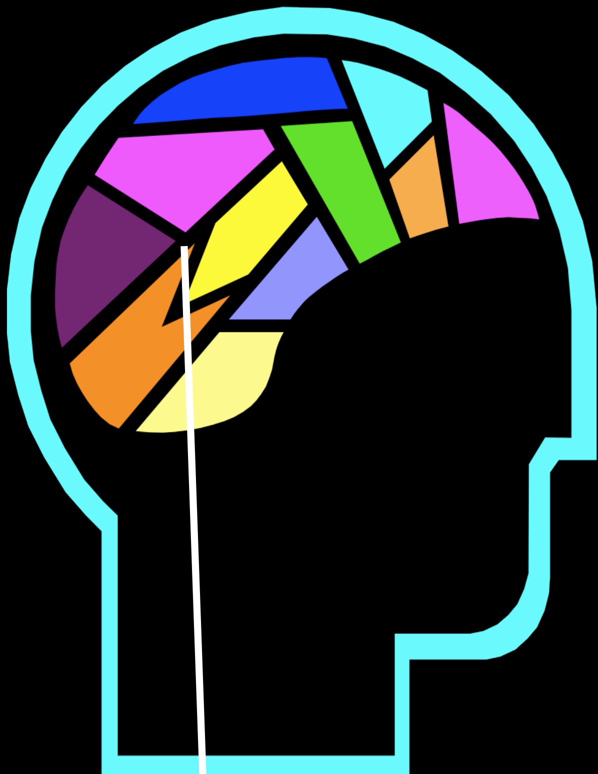
Draw simple pictures.

Try out small examples of the problem: What happens for n=1? n=2?



Novice

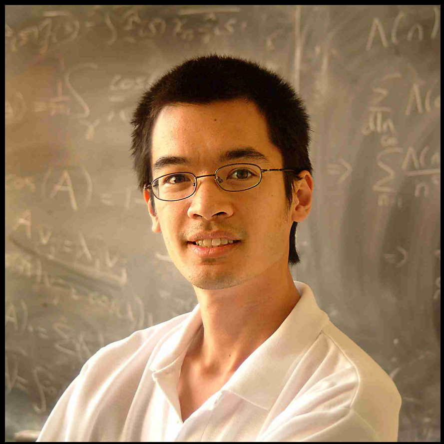
Expert



- The better the problem solver, the less brain activity is evident. The real masters show almost no brain activity!

Simple and to the point

"I don't have any magical ability…I look at the problem, and it looks like one I've already done. When nothing's working out, then I think of a small trick that makes it a little better. I play with the problem, and after a while, I figure out what's going on."



Terry Tao;

Fields Medalist, considered to be the best problem solver in the world.

Examples - I

What is the negation of the statement "If I drive then I am sober."

Solution: Consider this as an implication.

P: I will drive.

Q: I am sober.

The statement is of the form P -> Q. An implication is essentially (not P) or Q, and its negation would be P and (not Q). The negation of the statement, in plain English, is "I will drive and I will not be sober." (Don't do that!)

Examples - II

Richard is a knight or a knave. Knights always tell the truth and only the truth; knaves always tell falsehoods and only falsehoods. Someone asks Richard, “Are you a knight?” He replies, “If I am a knight then I’ll eat my hat.” Must Richard eat his hat?

Predicates

$$P(x) = x \text{ is taller than Jordan.}$$

- A predicate is a **propositional function**.
- When the function takes a specific value (or values), it becomes a proposition and has a truth value (true or false).

$P(\text{James}) =$ James is taller than Jordan; can be true or false.

Another predicate

$$P(x) = \{x + 4 > 10\}$$

True for all real numbers or integers greater than 6.

As is the case with normal functions, the domain for a predicate needs to be clearly defined.

Quantifiers

- Predicates are valid over a domain, and values from the domain change the predicate into a proposition.
- Another way to obtain a proposition from a predicate is through quantifiers that operate on the domain.

$$P(x) = \{x > 0\}$$

If the domain is all positive real numbers, then the proposition is always true.

$$\forall x, P(x)$$

For all valid *x*, *P(x)* is true.

The universal quantifier

- The universal quantifier of a predicate *P(x)* is the proposition "*P(x)* is true for all *x* in the universe of discourse."
- The universe of discourse is the domain of interest.

$$\forall x, P(x)$$

True if and only if *P(x)* is true for all *x*.
False if there exists at least one *x* such that *P(x)* is false.

The universal quantifier

$$P(x) = \{x \text{ is a student in the ECE department.}\}$$

Suppose that the domain is the set of students enrolled in EECE 320.

$$\forall x, P(x)?$$

The existential quantifier

- The existential quantifier of a predicate *P(x)* is the proposition "*P(x)* is true for **at least one** value of *x* in the universe of discourse."
- The universe of discourse is the domain of interest.

$$\exists x, P(x)$$

True if and only if *P(x)* is true for at least one value of *x*.
False if there exists no value of *x* such that *P(x)* is true.

The existential quantifier

$$P(x) = \{x \text{ is a student in the CS department.}\}$$

Suppose that the domain is the set of students enrolled in EECE 320.

$$\exists x, P(x)?$$

Some facts about quantifiers

- Suppose the universe of discourse is a finite set $\{x_1, x_2, \ldots, x_n\}$ then:

$$\exists x, P(x) \equiv P(x_1) \vee P(x_2) \vee \cdots \vee P(x_n)$$

$$\forall x, P(x) \equiv P(x_1) \wedge P(x_2) \wedge \cdots \wedge P(x_n)$$

$$\forall x, P(x) \rightarrow \exists x, P(x)$$

More examples with quantifiers

- *L(x)* = *x* is a lion.
- *F(x)* = *x* is a fierce creature.
- *C(x)* = *x* drinks coffee.
- Let the universe *U* be the set of all creatures.

All lions are fierce: $\forall x(L(x) \to F(x))$

Some lions do not drink coffee: $\exists x(L(x) \wedge \neg C(x))$

Some fierce creatures do not drink coffee: $\exists x(F(x) \wedge \neg C(x))$

Negating quantified predicates

$$\neg(\forall x, P(x))$$

$\updownarrow$

not (*P(x)* is true for all *x*)

$\updownarrow$ **not the same as (*P(x)* is not true for any *x*).**

For some *x*, *P(x)* is false

$\updownarrow$

$$\exists x, \neg P(x)$$

De Morgan's Laws for quantifiers

What would the negation of the existential quantifier yield?

Free and bound variables

- A variable is **bound** if its value is known or if the variable is within the scope of some quantifier.
- A variable that is not bound is **free**.
- In $P(x)$, x is free. If we consider the proposition $P(b)$, x is bound to the value b.
- Similarly, x is bound in $\forall x, P(x)$.
- A predicate becomes a proposition only when all variables are bound.

Multiple variables

- Predicates need not be restricted to one variable.
- $P(x, y)$ = "$x > y$" (defined over the space of all integers x and y)
  - Multiple variables permit multiple quantifiers.
  - $\forall x(P(x,y))$ is not a proposition; y is free.
  - $\forall x \forall y(P(x,y))$ is a contradiction -- always false.
  - $\forall x \exists y(P(x,y))$ is a tautology -- always true.

Interpreting multiple quantifiers

- $\forall x \forall y(P(x,y))$: $P(x,y)$ is true for every combination of x and y values.
- $\exists x \exists y(P(x,y))$: $P(x,y)$ is true for at least one combination of x and y values.
- $\exists x \forall y(P(x,y))$: There is at least one value of x (call it a) such that $P(a,y)$ is true for all values of y.

Start with the outermost (left-most) quantifier and work inwards.

Methods of Proof

Rules of inference

Rule	Tautology	Name
p $\therefore p \lor q$	$p \to (p \lor q)$	Addition
$p \land q$ $\therefore p$	$(p \land q) \to p$	Simplification
p $p \to q$ $\therefore q$	$[p \land (p \to q)] \to q$	Modus ponens
$\neg q$ $p \to q$ $\therefore \neg p$	$[\neg q \land (p \to q)] \to \neg p$	Modus tollens
$p \to q$ $q \to r$ $\therefore p \to r$	$[(p \to q) \land (q \to r)] \to (p \to r)$	Hypothetical syllogism
$p \lor q$ $\neg p$ $\therefore q$	$[(p \lor q) \land \neg p] \to q$	Disjunctive syllogism

Fallacies

- How proofs can go wrong!
- Example: Affirming the conclusion.
    - If I am Katarina Witt then I skate fast.
    - I skate fast.
    - Therefore: I am Katarina Witt.
- Example: Denying the hypothesis.
    - If it rains then it is cloudy.
    - It is not raining.
    - Therefore: It is not cloudy.

Methods of Proof - 1

- Vacuous proof
  - To show that “P implies Q” is true, it is sufficient to show that P is always false.
  - P: Elephants are small.
  - Q: Global warming is man-made.
- Trivial proof
  - To show that “P implies Q” is true, it is sufficient to show that Q is always true.
  - P: Elephants are small.
  - Q: The Earth is not flat.

Methods of Proof - 2

- Direct proof
  - To show that "P implies Q" is true, assume that P is true and use the inference rules to show that Q is true.
  - Common method of proof.
  - Example: If $n \equiv 3 \bmod 4$ then $n^2 \equiv 1 \bmod 4$.

Methods of Proof - 3

- Indirect proof
  - To show that “P implies Q”, we can show that the equivalent statement “not Q implies not P” is true.
- Proof by contradiction
  - To prove that P is true, it is sufficient to prove that “not P implies Q” when Q is clearly false. (Similar to the disjunctive syllogism.)
  - Example: Prove that $\sqrt{2}$ is irrational.

Methods of Proof - 4

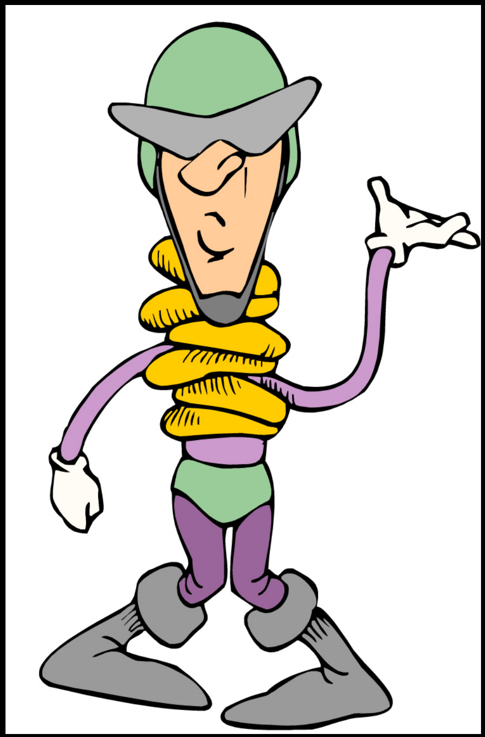
- **Proof by cases**

Suppose $p \equiv (p_1 \vee p_2 \vee \cdots \vee p_n)$, then
$p \rightarrow q$ is the same as
$(p_1 \rightarrow q) \wedge (p_2 \rightarrow q) \wedge \cdots \wedge (p_n \rightarrow q)$.

Methods of Proof - 5

- Constructive and non-constructive proofs
  - **Constructive proof**: To show that $\exists x(P(x))$ we can find x such that $P(x)$ is true.
    - Example: For any n (natural number), show that there exist n consecutive composite numbers.
  - **Non-constructive proof**: We may be able to show that $\exists x(P(x))$ even without finding a specific x.
    - Example: For any natural number n, show that there exists a prime number p such that $p > n$.

Wrap up



- Predicate logic
  - Quantifiers
  - Multiple variables
- Rules of inference
- Methods of proof
  - Fallacies (beware!)
  - Various proof techniques