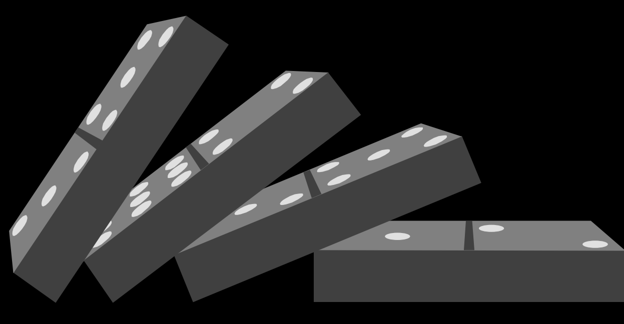
Discrete Structures & Algorithms

Mathematical Induction

EECE 320 // UBC

Dominoes

Domino Principle: Line up any number of dominos in a row; knock the first one over and they will all fall.



Dominoes numbered 1 to n

F_k = "The k^{th} domino falls"

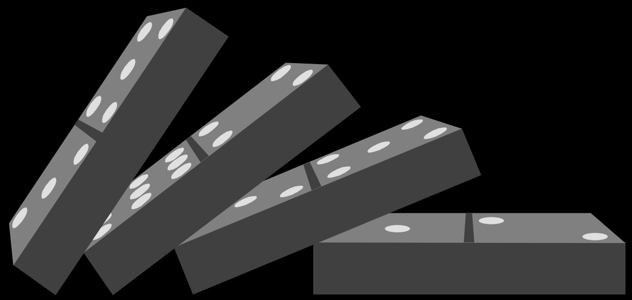
If we set them up in a row then each one is set up to knock over the next:

For all $1 \leq k < n$:

$$F_k \Rightarrow F_{k+1}$$

$F_1 \Rightarrow F_2 \Rightarrow F_3 \Rightarrow \ldots$

$F_1 \Rightarrow$ All Dominoes Fall



Dominoes numbered 1 to n

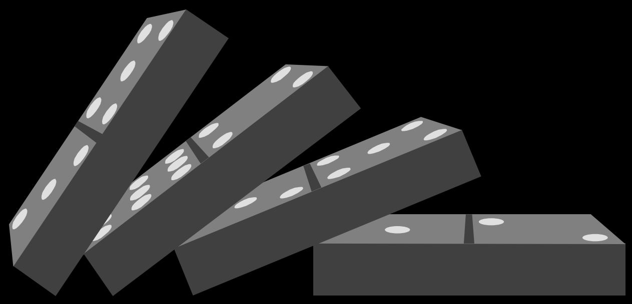
F_k = "The k^{th} domino falls"

$\forall k, 0 \leq k < n-1$:

$$F_k \Rightarrow F_{k+1}$$

$F_0 \Rightarrow F_1 \Rightarrow F_2 \Rightarrow \ldots$

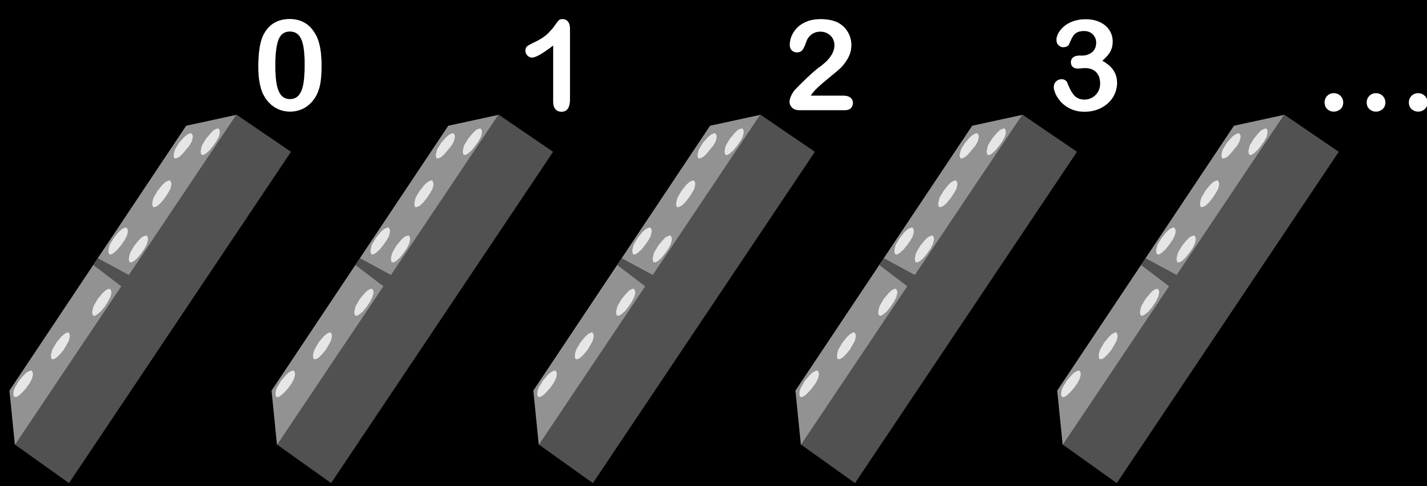
$F_0 \Rightarrow$ All Dominoes Fall

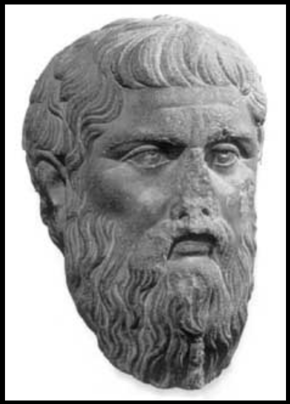


The natural numbers

N = { 0, 1, 2, 3, . . . }

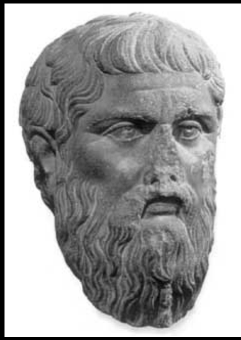
One domino for each natural number:





Plato: The Domino Principle works for an infinite row of dominoes.

Aristotle: Never seen an infinite number of anything, much less dominoes.



Plato's Dominoes
One for each natural number

Theorem: An infinite row of dominoes, one domino for each natural number.
Knock over the first domino and they all will fall

Proof:
Suppose they don't all fall. Let $k > 0$ be the lowest numbered domino that remains standing. Domino $k-1 \geq 0$ did fall, but $k-1$ will knock over domino k. Thus, domino k must fall and remain standing. Contradiction.

Mathematical Induction

statements proved instead of dominoes fallen

Infinite sequence of dominoes

F_k = "domino k fell"

Infinite sequence of statements: S_0, S_1, …

F_k = "S_k proved"

Establish: 1. F_0

2. For all k, $F_k \Rightarrow F_{k+1}$

Conclude that F_k is true for all k

Inductive proofs

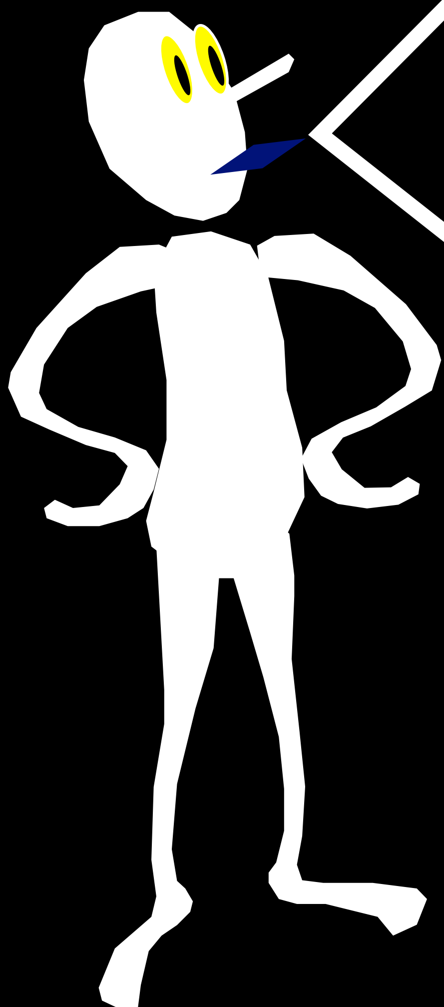
To prove $\forall k \in N, S_k$

Establish "Base Case": S_0

Establish that $\forall k, S_k \Rightarrow S_{k+1}$

$\forall k, S_k \Rightarrow S_{k+1}$

- Assume hypothetically that S_k for any particular k;
- Conclude that S_{k+1}

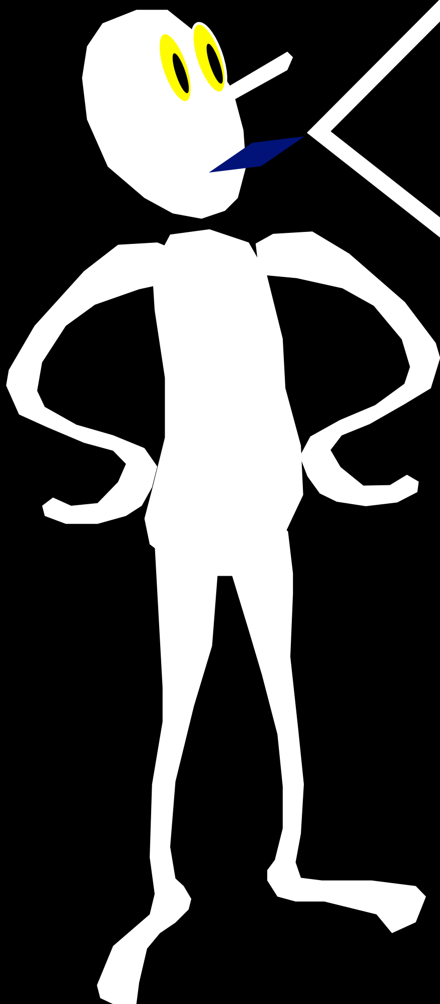


Theorem?

The sum of the first n odd numbers is n^2

Check on small values:

1	= 1
1+3	= 4
1+3+5	= 9
1+3+5+7	= 16



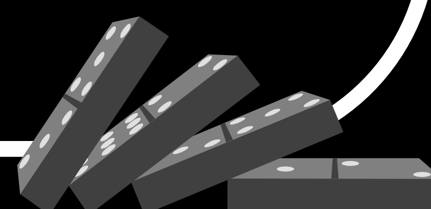
Theorem?

The sum of the first n **odd** numbers is n^2

The k^{th} odd number is $(2k - 1)$, when $k > 0$

S_n is the statement that:

"$1+3+5+(2k-1)+\ldots+(2n-1) = n^2$"



Establishing that $\forall n \geq 1\ S_n$

S_n = "$1 + 3 + 5 + (2k-1) + . . +(2n-1) = n^2$"

Base Case: S_1

Domino Property:

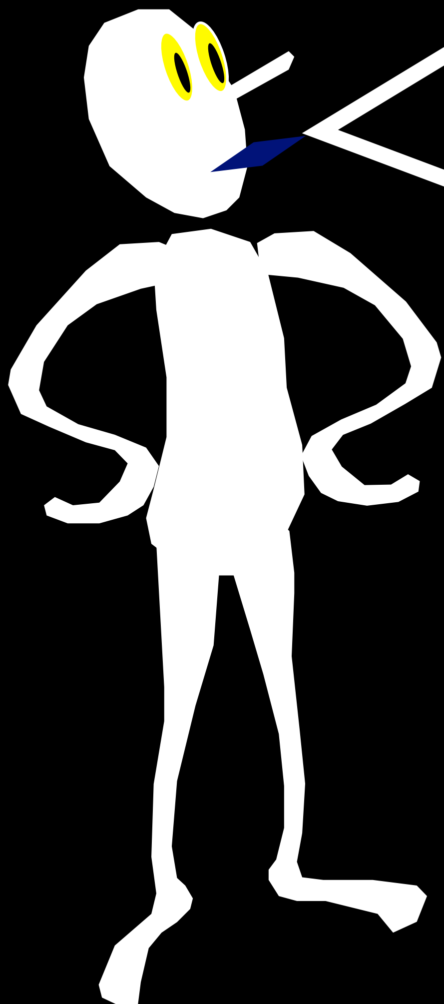
Assume "Induction Hypothesis": S_k

That means:

$1+3+5+\ldots+ (2k-1) \qquad = k^2$

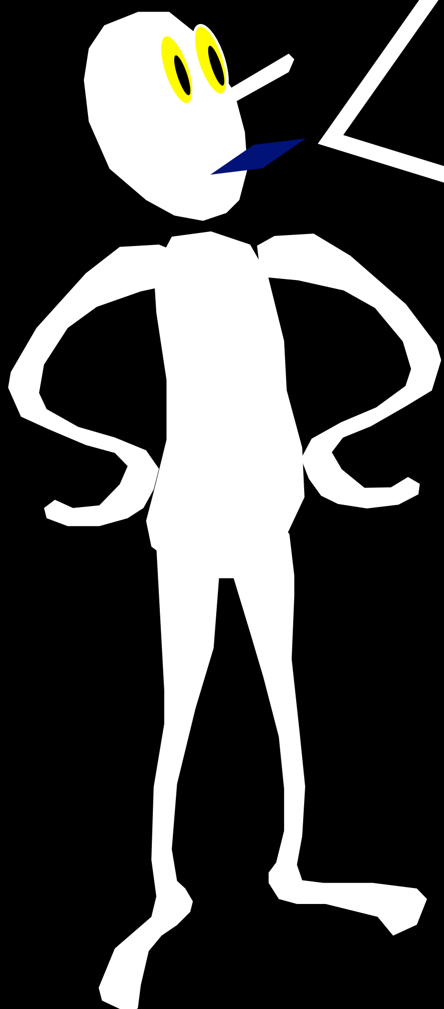
$1+3+5+\ldots+ (2k-1)+(2k+1) \qquad = k^2 +(2k+1)$

Sum of first k+1 odd numbers = $(k+1)^2$



Theorem

The sum of the first n odd numbers is n^2



Primes:

A natural number n > 1 is a **prime** if it has no divisors besides 1 and itself

Note: 1 is not considered prime

Theorem?

Every natural number > 1 can be factored into primes

S_n = "n can be factored into primes"

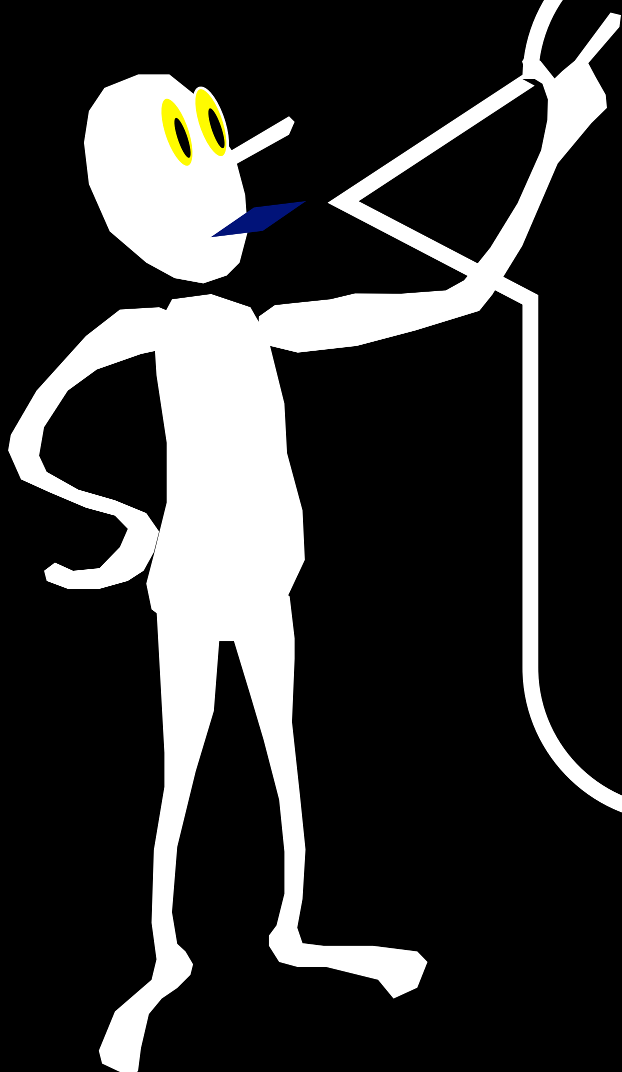
Base case:
2 is prime $\Rightarrow S_2$ is true

How do we use the fact:

S_{k-1} = "k-1 can be factored into primes"

to prove that:

S_k = "k can be factored into primes"



This illustrates a technical point about mathematical induction

Theorem?

Every natural number > 1 can be factored into primes

A different approach:

Assume 2,3,…,k-1 all can be factored into primes

Then show that k can be factored into primes

“All previous” induction

To prove $\forall k, S_k$

Establish Base Case: S_0

Establish Domino Effect:

Assume $\forall j<k, S_j$

use that to derive S_k

Also called "Strong Induction"

Establish Domino Effect:

Assume $\forall j<k, S_j$

use that to derive S_k

"All previous" induction

Repackaged As Standard Induction

Define $T_i = \forall j \leq i, S_j$

Establish Base Case: S_0

Establish Domino Effect:

Let k be any number

Assume $\forall j<k, S_j$

Prove S_k

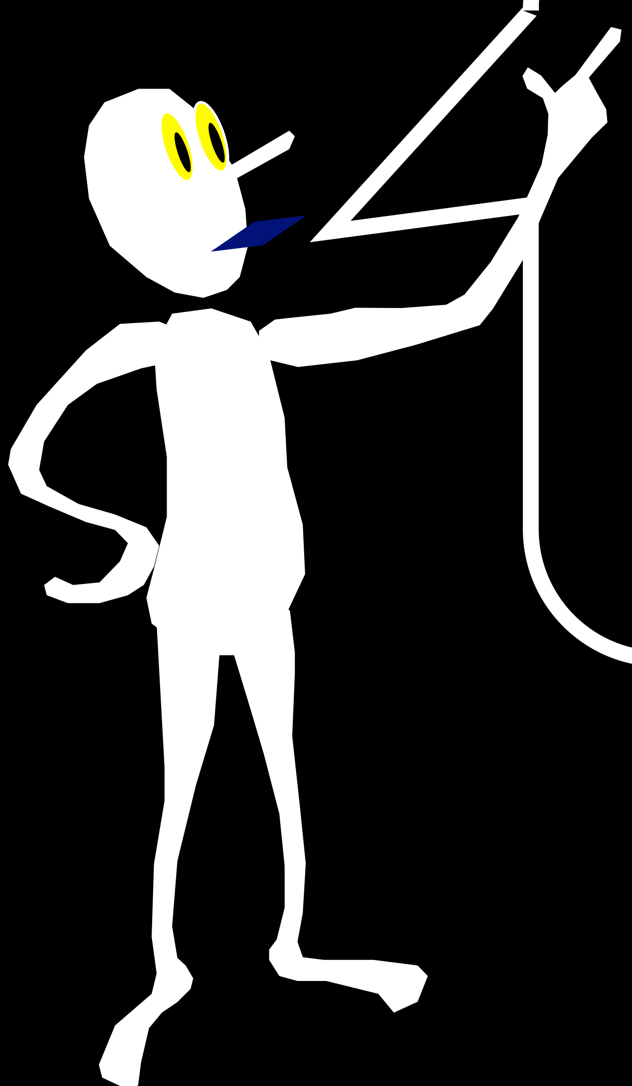
Establish Base Case T_0

Establish that $\forall k, T_k \Rightarrow T_{k+1}$

Let k be any number

Assume T_{k-1}

Prove T_k



And there are more ways to do inductive proofs

Method of Infinite Descent

Rene Descartes

Show that for any counter-example you find a smaller one

If a counter-example exists there would be an infinite sequence of smaller and smaller counter examples

Theorem:

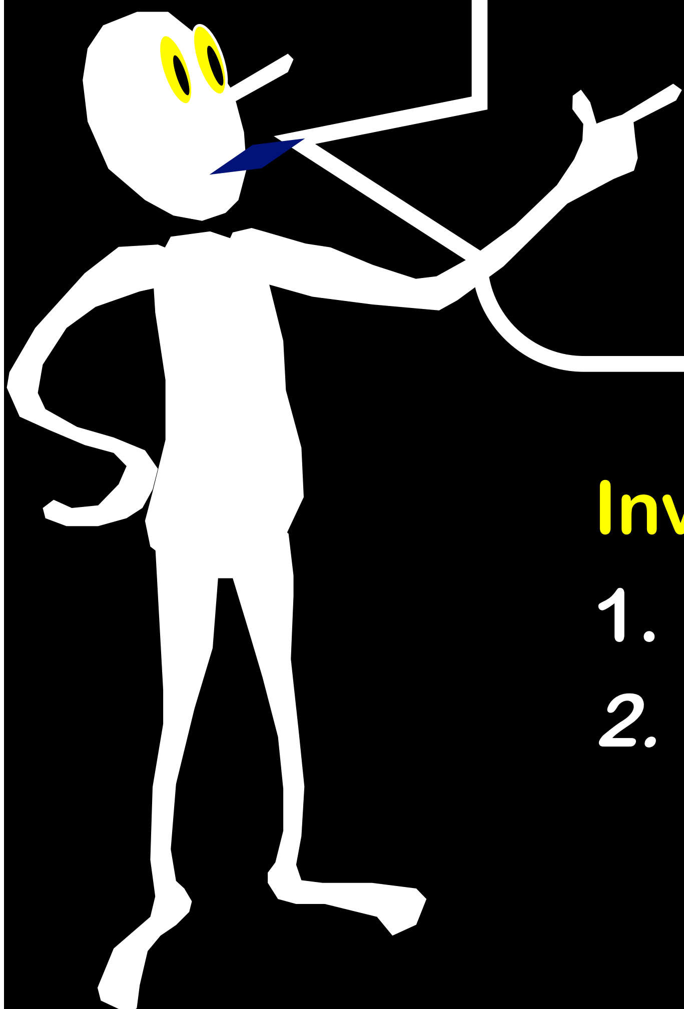
Every natural number > 1 can be factored into primes

Let n be a counter-example

Hence n is not prime, so n = ab

If both a and b had prime factorizations, then n would too

Thus a or b is a smaller counter-example



Yet another way of packaging inductive reasoning is to define "invariants"

Invariant (n):

1. Not varying; constant.

2. Mathematics. **Unaffected by a designated operation, as a transformation of coordinates.**

Invariant (n):

3. *Programming.* A rule, such as the ordering of an ordered list, that applies throughout the life of a data structure or procedure. Each change to the data structure maintains the correctness of the invariant

Invariant Induction

Suppose we have a time varying world state: W_0, W_1, W_2, …

Each state change is assumed to come from a list of permissible operations. We seek to prove that statement S is true of all future worlds

Argue that S is true of the initial world

Show that if S is true of some world – then S remains true after one permissible operation is performed

Odd/Even Handshaking Theorem

At any party at any point in time define a person's parity as ODD/EVEN according to the number of hands they have shaken

Statement: The number of people of odd parity must be even

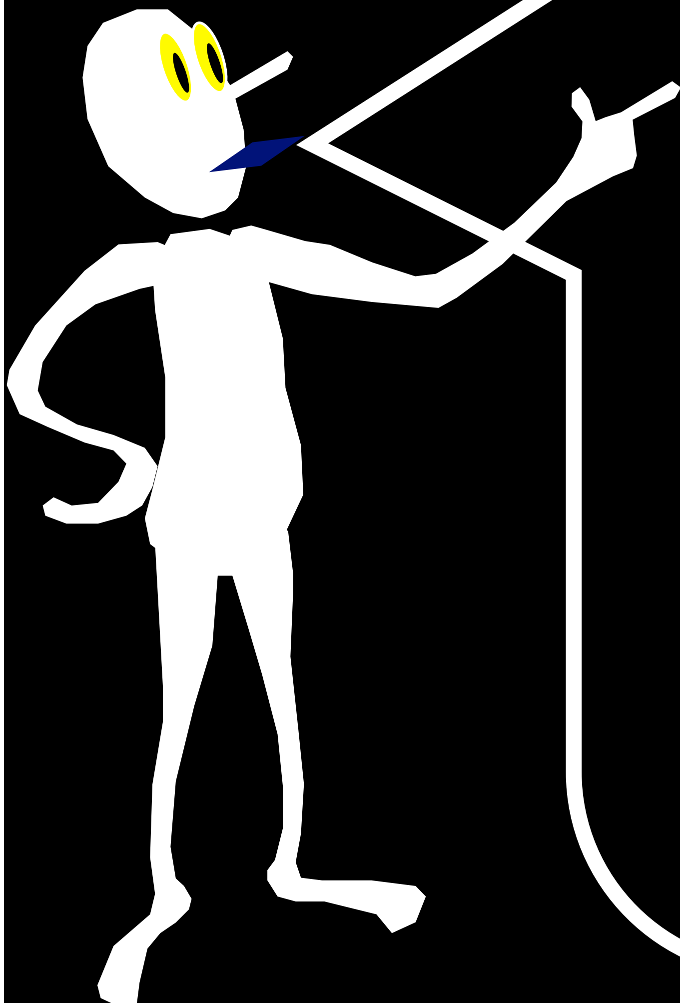
Statement: The number of people of odd parity must be even

Initial case: Zero hands have been shaken at the start of a party, so zero people have odd parity

Invariant Argument:

If 2 people of the same parity shake, they both change and hence the odd parity count changes by 2 – and remains even

If 2 people of different parities shake, then they both swap parities and the odd parity count is unchanged



Inductive reasoning is the high level idea

"Standard" Induction

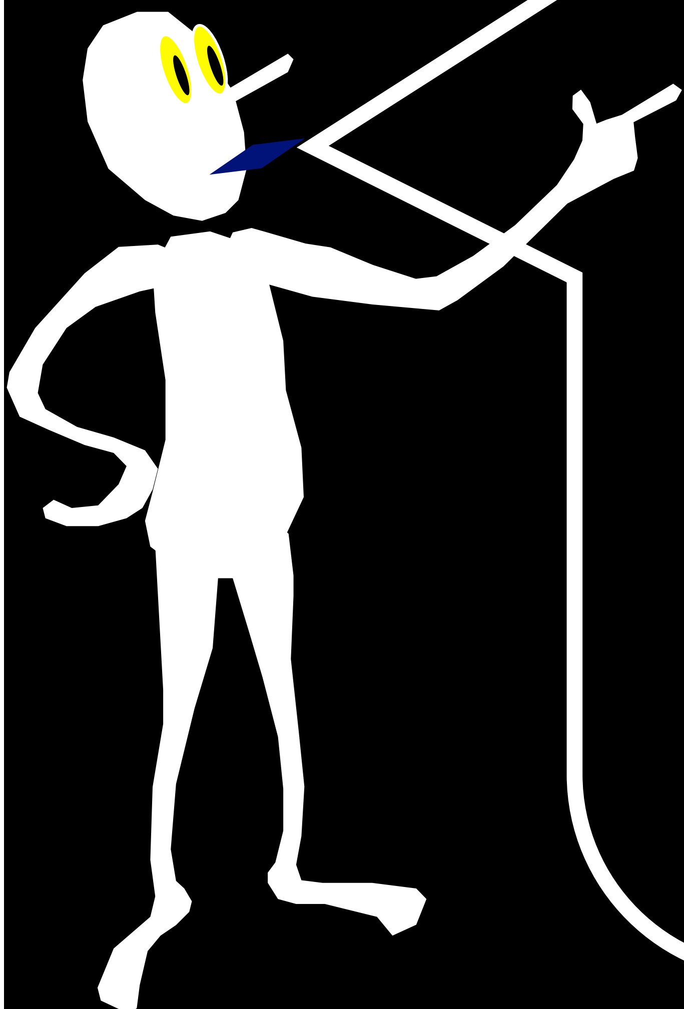
"All Previous" Induction

"Least Counter-example"

"Invariants"

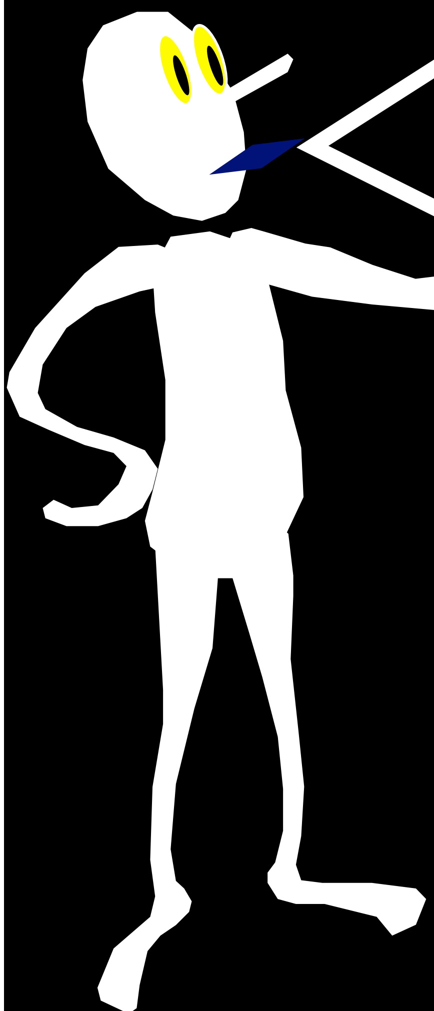
all just

different packaging



Induction is also how we can define and construct our world.

So many things, from buildings to computers, are built up stage by stage, module by module, each depending on the previous stages.



The essential elements of mathematical induction are:

- The proposition of interest
- The base case
- The inductive step

Remember these elements and you can prove many a result.