Some notes on counting with polynomials and combinatorial arguments

1 Counting with polynomials

Polynomials and their products provide very general – and powerful – methods to count a variety of sets. In particular, we will examine how to count the number of *non-negative integer* solutions to the equation

$$x_1 + x_2 + \cdots + x_n = m,$$

subject to a few constraints.

We will start by assuming only that each $x_i \geq 0$ and $m \geq 0$. This type of equation then represents that number of ways to divide up m bars of gold among n pirates with the possibility that some pirates may receive nothing at all. It is all identical to questions involving n identical bins and m identical balls: In how many ways can we put balls into bins?

An approach to solving this type of problem is to proceed as follows. Let us suppose we are dividing m bars of gold among n pirates. We may assume that the bars of gold are lined up, and we simply throw some dividers into the line to partition the bars of gold into n partitions.

Consider the specific example of $m = 6$ and $n = 3$. We could line up the bars of gold as GGGGGG. If | represents a partition, then one possible division of the gold among the three pirates is G|GGGG|GG, which means that the first pirate gets 1 bar of gold, the second pirate gets 4 bars of gold and the third pirate gets 2 bars of gold. A partition that looks like G||GGGGG would simply mean that the second pirate gets nothing at all. Notice that every possible partitioning of the gold can be represented by a string of length 8 made up of 6 Gs and 2 |s. We know that there are $\binom{8}{2} = 28$ such strings – we count by merely selecting the positions for the two dividers – and that is the number of ways to partition 6 bars of gold among 3 pirates.

In general, for m bars of gold and n pirates, there are $\binom{n+m-1}{n-1}$ possible allocations of gold. And in fact, that is the number of integer solutions to the equation $x_1 + \cdots + x_n = m$ such that each $x_i \geq 0$ and $m \geq 0$.

We shall, however, obtain the same result in a different fashion to convey the use of polynomials.

We want to find the number of integer solutions to $x_1 + \cdots + x_n = m$ such that each $x_i \geq 0$. Let us consider the polynomial $(x^0 + x^1 + x^2 + \dots)^n$. (Note: The x in this polynomial is a dummy variable that is not to be confused with x_i.) Now, we will let $(x^0 + x^1 + \dots)$ represent the value of x_i. An easy way to reason about this is an OR operation. We are saying that x_i is 0 or 1 or 2 or .... Now,

$$(x^0 + x^1 + \dots)^n = \sum_{i=0}^{\infty} c_k x^k,$$

if we expanded out. We let c_k denote the *coefficient* of x^k on the right side of the above equality. Notice that the coefficient of x^k represents the number of ways to achieve a sum of k using n variables. For instance, the $x_1 + \cdots + x_n = 0$ is possible only when each $x_i = 0$, which is in exactly one way. Therefore $c_0 = 1$. We are therefore interested in the coefficient of x^m in the expansion.

How do we determine the coefficient of x^m? We can make use of the fact that $x^0 + x^1 + \dots$ is an infinite geometric progression, and we let its sum be $\frac{1}{1-x} = (1-x)^{-1}$. (This is under the assumption that $0 < x < 1$, which we are free to make because x is a dummy variable.)

How do we expand $(1-x)^{-n}$ (because that is the function of interest)? We use Taylor's Series Expansion. For any function $f(x)$, the function can be expanded about a point a as follows:

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \dots.$$

For $f(x) = (1-x)^{-n}$ and expanding about $a = 0$, we get

$$(1-x)^{-n} = 1 + (n)\frac{(1-0)^{-(n+1)}}{1!}x + n(n+1)\frac{(1-0)^{-(n+2)}}{2!}x^2 + \cdots = \sum_{k=0}^{\infty} \binom{n+k-1}{k} x^k.$$

We are really interested in the coefficient of x^m in the expansion, and we now observe that the coefficient of x^m is $\binom{n+m-1}{m} = \binom{n+m-1}{n-1}$.

The earlier method for obtaining this result may have appeared easier. So, why this method? This is useful when we want to count the number of solutions for

$$x_1 + \cdots + x_n = m$$

when $0 \leq x_i \leq r$. In this case, the approach used for dividing the gold bars does not work because we are not able to ensure, in that method, that there is a minimum or maximum distance between the dividers we insert.

Using polynomials we can answer this question more easily. We are really interested in the coefficient of x^m in the expansion of $(x^0 + \cdots + x^r)^n$. Using the sum of a finite geometric progression, we know that we are really interested in the coefficient of x^m in the expansion of

$$\left(\frac{1-x^{r+1}}{1-x}\right)^n = (1-x^{r+1})^n(1-x)^{-n}.$$

We know that we can expand $(1-x^{r+1})^n$ using the binomial theorem, and we can expand $(1-x)^{-n}$ as we have seen earlier. We can then determine what the coefficient of interest is.

Example. How many integer solutions are there to the equation

$$x_1+x_2+x_3+x_4 = 15$$

if $0 \leq x_i \leq 6$.

The number of solutions can be obtained by identifying the coefficient of x^{15} in the expansion of $(1-x^7)^4(1-x)^{-4}$. Using the binomial theorem

$$(1-x^7)^4 = 1-4x^7+6x^{14}-4x^{21}+x^{28}.$$

Further, we know that the expansion of $(1-x)^{-4}$ is

$$(1-x)^{-4} = \sum_{k=0}^{\infty}\binom{3+k}{k}x^k.$$

The coefficient of x^{15} in the expansion of $(1-x^7)^4(1-x)^{-4}$ can be obtained by considering the following pairs of coefficients: $\{$the coefficient of x^0 in the expansion of $(1-x^7)^4$ and the coefficient of x^{15} in the expansion of $(1-x)^{-4}\}$, $\{$the coefficient of x^7 in the expansion of $(1-x^7)^4$ and the coefficient of x^8 in the expansion of $(1-x)^{-4}\}$, $\{$the coefficient of x^{14} in the expansion of $(1-x^7)^4$ and the coefficient of x^1 in the expansion of $(1-x)^{-4}\}$. There is no other way to obtain x^{15} in the expansion of $(1-x^7)^4(1-x)^{-4}$. Using this knowledge we obtain

$$c_{15} = (1\times\binom{18}{15})+(-4\times\binom{11}{8})+(6\times\binom{4}{1}) = 816-660+24 = 180.$$

The number of integer solutions to the given equation subject to the constraints is 180.

Remarks. Counting using polynomials is an extremely powerful technique. Also, determining the number of integer solutions to an equation is an excellent abstraction for many problems, such as the balls and bins problem and the pirates and gold problem.

2 Combinatorial arguments

A combinatorial argument is a method of proof that is often used to prove certain identities involving combinatorial objects. Some identities are not easily amenable to algebraic manipulation, or it may be simpler to use a combinatorial argument. The core of a combinatorial argument lies in demonstrating that a set of objects is being counted in two different ways. It is best to illustrate this idea with some examples.

Example 1. Give a combinatorial argument to show that

$$\sum_{k=1}^{n} k\binom{n}{k} = n2^{n-1}.$$

Solution. To solve this problem, we first attempt to identify what sets might be getting counted. The right side of the identity is easier to start with because it does not involve a summation. Recall that the number of ways to select a subset from a set of n objects is 2^n; for each object, you can choose to include or exclude it from a subset. Also, the number of ways of selecting one object from a set of size n is n. In this question, it does appear that we are doing both: selecting one object from a set of size n and then picking a subset from a set of size $n-1$ resulting in $n2^{n-1}$ choices.

We can advance the argument by creating an analogy. Imagine that we are selecting a committee from a set of n people. We also want to choose a leader for the committee. We require that a committee have a leader but it need not include anyone else. In how many ways can we choose such a committee?

One approach is to be oblivious to the size of the committee. Let us pick one person to lead the committee (there are n ways to do so) and then pick a subset (possibly the empty set) of the remaining $n-1$ candidates to comprise the rest of the committee. There are exactly $n2^{n-1}$ ways to choose a committee! Here we do not explicitly care about the size of the committee.

Another approach may be to partition the possibilities based on the size of the committee. If the size of the committee is k, we can first pick the committee members in $\binom{n}{k}$ ways and then select a leader in k ways. That means that there are $k\binom{n}{k}$ ways to pick a committee of size k and its leader. The smallest possible committee has to have at least one member (the leader) and the largest committee can have at most n members. Because there is no restriction on the size of the committee, all sizes are possible and therefore $\sum_{k=1}^{n} k\binom{n}{k}$ is also the number of ways to choose a committee, and its leader, from a pool of n people. But this is exactly what we did earlier, except that we used a different procedure that did not partition the possibilities based on the size of the committee.

Thus we have shown that

$$\sum_{k=1}^{n} k\binom{n}{k} = n2^{n-1}.$$

Example 2. Give a combinatorial argument to show that

$$\sum_{k=1}^{n} k\binom{n}{k}^2 = n\binom{2n-1}{n-1}.$$

Solution. We can proceed in a manner similar to the previous example. Here the right side of the identity suggests that we are selecting an object from n possible objects and another $n-1$ objects from a set of $2n-1$ objects. In all, we are selecting n objects.

We are likely to leap ahead and say: "Hey, maybe we are selecting a committee of size n from a set of $2n$ people and selecting its leader." In that case, we should ask ourselves why there are only n choices for the leader and not $2n$. This suggests that the leader of the committee may be restricted to a special set of n people. How could we explain that?

To proceed further, consider selecting a committee of size n from a group of n ECE and n CS professors. The requirement is that the leader of the committee be an ECE professor. If we elaborate on this situation further and make use of the fact that $\binom{n}{k} = \binom{n}{n-k}$, we will find that the left side and the right side of the identity are indeed the number of ways of choosing such a committee.

The two examples above illustrate the principle of using a combinatorial argument but there is much verbiage that can be skipped. To provide a brief and precise argument, follow the next example.

Example 3. Prove that

$$\sum_{k=0}^{r} \binom{n+k}{k} = \binom{n+r+1}{r}$$

using a combinatorial argument.

Proof. Consider a set of $n+r+1$ integers. Without loss of generality, we can assume that the integers are $\{1,2,3,\ldots,n,n+1,\ldots,n+r+1\}$. The number of ways to choose a set of $n+1$ integers from this set of $n+r+1$ integers is $\binom{n+r+1}{n+1} = \binom{n+r+1}{r}$.

We can also partition the subsets of size $n+1$ on the basis of the maximum element in the subset. Because we are selecting a subset with $n+1$ integers from this set of $n+r+1$ integers, we have to choose either $n+1$ or a larger integer. If $n+1$ is the maximum element in the subset, then there is only 1 way to choose such a subset. If $n+2$ is the largest element of the subset then the number of such subsets is $\binom{n+1}{n} = \binom{n+1}{1}$. If $n+3$ is the largest element of the subset then there are $\binom{n+2}{n} = \binom{n+2}{2}$ such subsets. In general, if $n+k+1$ is the largest element of the subset then there are $\binom{n+k}{k}$ such subsets. The maximum element of a subset of size $n+1$ is no smaller than $n+1$ and no larger than $n+r+1$, and we are interested in all size $n+1$ subsets of the original set of size $n+r+1$. This is indeed what the left side of the proposition represents.

The left side and the right side represent different ways to count the same subsets and therefore the proposition is true.

$\square$

The textbook suggests a different approach to the above example (Rosen, 6th edition, p369, q27). There are numerous combinatorial arguments for the same identity, and it requires some creativity to develop a combinatorial argument.