$\boxed{\text{A } \textit{\textbf{set}} \text{ is an unordered collection of objects, called its } \textit{\textbf{elements}}.}$

$$\{\text{Groucho, Chico, Harpo, Zeppo, Gummo}\}$$
$$=$$
$$\{\text{Chico, Groucho, Harpo, Gummo, Zeppo}\}$$
$$=$$
$$\{\text{Groucho, Gummo, Harpo, Chico, Harpo, Groucho, Groucho, Zeppo}\}$$

Some famous sets:

- The *suits* $\{\spadesuit, \heartsuit, \diamondsuit, \clubsuit\}$
- The *non-negative integers* (or *natural numbers*) $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$
- The *integers* $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$
- The *real numbers* $\mathbb{R}$
- The *empty set* $\varnothing = \{\,\}$
- $\{\varnothing\} = \{\{\,\}\}$ is a set with one element, and thus is *not* the empty set. The unique element of $\{\varnothing\}$ is the empty set.
- $\{\varnothing, \mathbb{N}, \mathbb{Z}\}$ is a set with three elements (which happen to be sets).

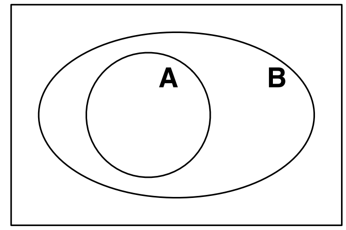
Set Notation

$\boldsymbol{x \in S}$ — x is an element of S

$\boldsymbol{x \notin S}$ — x is *not* an element of S.

$\neg(x \in S)$

$\boldsymbol{A \subseteq B}$ — A is a ***subset*** of B.



Every element of A is also an element of B.

$\forall x\,(x \in A \to x \in B)$

$\boldsymbol{A \supseteq B}$ — A is a ***superset*** of B.

$B \subseteq A$

$\forall x\,(x \in B \to x \in A)$

$\boldsymbol{A = B}$ — A and B are ***equal***.

Every element of A is also an element of B and vice versa.

$(A \subseteq B) \wedge (B \subseteq A)$.

$\forall x\,(x \in A \leftrightarrow x \in B)$

$\boldsymbol{A \subset B}$ — A is a ***proper subset*** of B.

$(A \subseteq B) \wedge (A \neq B)$.

$(A \subseteq B) \wedge (A \not\supseteq B)$.

$\forall x\,(x \in A \to x \in B) \;\wedge\; \exists y\,(y \in B \wedge y \notin A)$

$\boldsymbol{A \supset B}$ — A is a ***proper superset*** of B.

$B \subset A$

$(B \subseteq A) \wedge (B \neq A)$.

$(B \subseteq A) \wedge (B \not\supseteq A)$.

$\forall x\,(x \in B \to x \in A) \;\wedge\; \exists y\,(y \in A \wedge y \notin B)$

- Harpo $\in$ {Groucho, Chico, Harpo, Gummo, Zeppo}

- Karl $\notin$ {Groucho, Chico, Harpo, Gummo, Zeppo}

- {Groucho, Chico, Harpo} $\subseteq$ {Groucho, Chico, Harpo, Gummo, Zeppo}

- {Groucho, Chico, Harpo} $\subset$ {Groucho, Chico, Harpo, Gummo, Zeppo}

- {Groucho, Chico, Harpo} $\subseteq$ {Groucho, Chico, Harpo}

- $\varnothing \subseteq$ {Groucho, Chico, Harpo}

  Proof: No matter what x is, $x \in \varnothing$ is false.
  Thus, $\forall x \ ((x \in \varnothing) \to$ {Groucho, Chico, Harpo}$)$ is *vacuously* true! $\square$

- $\varnothing \notin$ {Groucho, Chico, Harpo}

- $\varnothing \in \{\varnothing,$ Ren, Stimpy$\}$

- $\varnothing \subset \{\varnothing,$ Ren, Stimpy$\}$

- $\{\varnothing,\ \{\varnothing\}\} \in \left\{\ \varnothing,\ \{\varnothing\},\ \{\{\varnothing\}\},\ \{\varnothing, \{\varnothing\}\}\ \right\}$

Proof of Set Containment

Theorem: $A \subseteq B$

Proof: Let x be an arbitrary element of A.

— Insert proof of $x \in B$ here —

We conclude that every element of A is also an element of B. $\square$

Proof of Set Equivalence

Theorem: $A = B$

Proof:

— Insert proof of $A \subseteq B$ here —

— Insert proof of $B \subseteq A$ here —

We conclude that A and B are the same set. $\square$

Describing/defining sets

1. List all the elements (only for finite sets)

   - $\{\spadesuit, \heartsuit, \diamondsuit, \clubsuit\}$
   - {Eenie, Meenie, Minie, Moe}
   - {Larry, Curly, Shemp, Curly Joe, Moe}
   - $\{\,\} = \varnothing$

2. List the elements 'implicitly' (but this is ambiguous)

   - $\{0, 1, 2, 3, 4, \ldots\}$
   - $\{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$
   - $\{2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37 \ldots\}$
   - $\{0, 2, 1, 5, 7, 3, 10, 4, 13, 15, 6, 18, 20, 8, 23, 9, 26, 28, 11, 31, 12, \ldots\}$

3. ***Set-builder notation:*** describe a property that defines the set.

   - $\boldsymbol{\{x \mid P(x)\}}$ = the set of all x such that $P(x)$ is true.
   - $\boldsymbol{\{x \in A \mid P(x)\}}$ = the set of all $x \in A$ such that $P(x)$ is true.
   - $\{x \mid x \text{ is prime}\}$
   - $\{x \mid x/2 \in \mathbb{Z}\} = \{x \mid \exists y \in \mathbb{Z}\, (x = 2y)\}$
   - $\{x \in CS173 \mid x \text{ is asleep}\}$

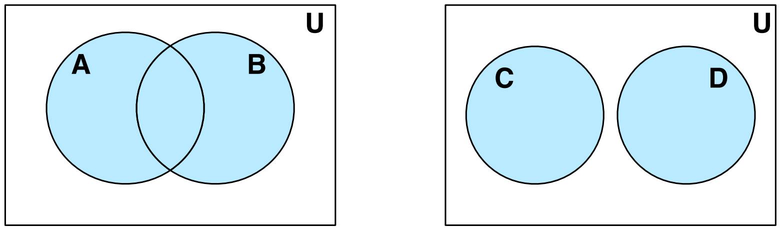
4. ***Recursively***

   - You can legally vote if your father or grandfather could legally vote.
   - $0 \in \mathbb{N}$, and for every $x \in \mathbb{N}$ we also have $x + 1 \in \mathbb{N}$.
   - $1 \in T$, and for every $x \in T$ we also have $2x \in T$.
   - $0 \in S$, and for every $x \in S$ we also have $x + 2\sqrt{x} + 1 \in S$.
   - Every axiom is a theorem, and any statement that can be logically inferred from theorems is a theorem.

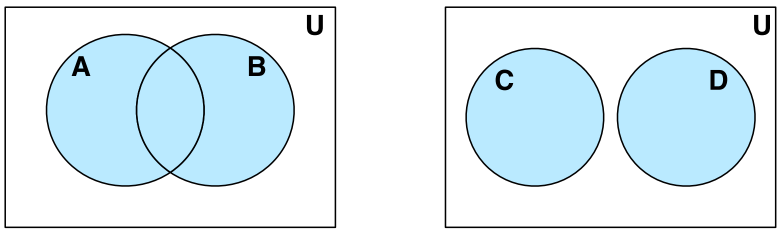
   Recursive set definitions are always *exclusive*: Anything that does not satisfy the recursive condition is not and element of the set.

5. ***Combining other sets:***

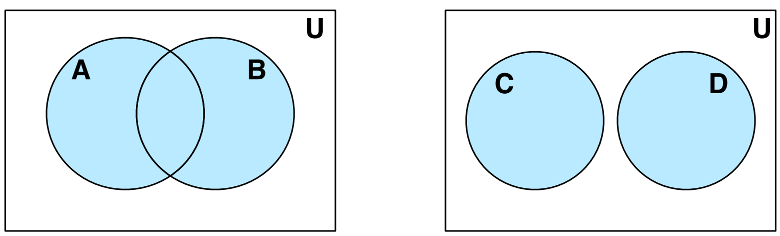
- ***Union*** $\boldsymbol{A} \cup \boldsymbol{B} = \{x \mid x \in A \ \vee \ x \in B\}$



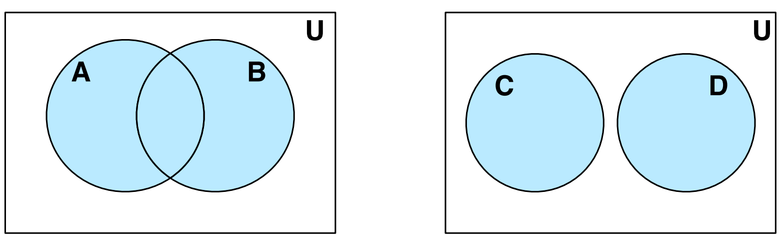
- ***Intersection*** $\boldsymbol{A} \cap \boldsymbol{B} = \{x \mid x \in A \ \wedge \ x \in B\}$



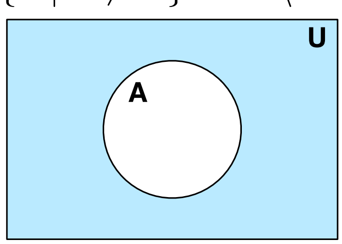
- ***Difference*** $\boldsymbol{A} \setminus \boldsymbol{B} = \{x \mid x \in A \ \wedge \ x \notin B\}$



- ***Symmetric difference*** $\boldsymbol{A} \oplus \boldsymbol{B} = \{x \mid x \in A \ \oplus \ x \in B\}$



- ***Complement*** $\overline{\boldsymbol{A}} = \{x \mid x \notin A\} = U \setminus A$



Suppose

$A = \{$Eenie, Meeine, Minie, Moe$\}$
$B = \{$Larry, Curly, Shemp, Moe$\}$
$C = \{$Homer, Ned, Willie, Krusty, Moe$\}$
$D = \{$Eenie, Krusty, Shemp$\}$

Then what are the following sets?

- $A \cup B =$
- $A \cap B =$
- $A \setminus B =$
- $C \cap D =$
- $(A \cap C) \cap D =$
- $(A \cup B) \setminus C =$
- $(A \cup B) \setminus D =$
- $B \oplus C =$

The 16 binary boolean functions

p	q	T	$p \vee q$	$p \leftarrow q$	p	$p \rightarrow q$	q	$p \leftrightarrow q$	$p \wedge q$
T	T	T	T	T	T	T	T	T	T
T	F	T	T	T	T	F	F	F	F
F	T	T	T	F	F	T	T	F	F
F	F	T	F	T	F	T	F	T	F

p	q	$p \uparrow q$	$p \oplus q$	$\neg q$	$p \nrightarrow q$	$\neg p$	$p \nleftarrow q$	$p \downarrow q$	F
T	T	F	F	F	F	F	F	F	F
T	F	T	T	T	T	F	F	F	F
F	T	T	T	F	F	T	T	F	F
F	F	T	F	T	F	T	F	T	F

The 16 binary set operations

A	B	U	$A \cup B$	$A \cup \overline{B}$	A	$\overline{A} \cup B$	B	$\overline{A \oplus B}$	$A \cap B$
T	T	T	T	T	T	T	T	T	T
T	F	T	T	T	T	F	F	F	F
F	T	T	T	F	F	T	T	F	F
F	F	T	F	T	F	T	F	T	F

A	B	$\overline{A \cap B}$	$A \oplus B$	$\overline{B}$	$A \setminus B$	$\overline{A}$	$B \setminus A$	$\overline{A \cup B}$	$\varnothing$
T	T	F	F	F	F	F	F	F	F
T	F	T	T	T	T	F	F	F	F
F	T	T	T	F	F	T	T	F	F
F	F	T	F	T	F	T	F	T	F

Don’t these look familiar?

Common logical equivalences

	Double negation: $\neg\neg p \iff p$	
$p \wedge T \iff p$	Identity	$p \vee F \iff p$
$p \wedge F \iff F$	Domination	$p \vee T \iff T$
$p \wedge p \iff p$	Idempotence	$p \vee p \iff p$
$p \wedge \neg p \iff F$	Negation	$p \vee \neg p \iff T$
$p \wedge (p \vee q) \iff p$	Absorption	$p \vee (p \wedge q) \iff p$
$p \wedge q \iff q \wedge p$	Commutativity	$p \vee q \iff q \vee p$
$(p \wedge q) \wedge r \iff p \wedge (q \wedge r)$	Associativity	$(p \vee q) \vee r \iff p \vee (q \vee r)$
$p \wedge (q \vee r) \iff (p \wedge q) \vee (p \wedge r)$	Distribution	$p \vee (q \wedge r) \iff (p \vee q) \wedge (p \vee r)$
$\neg(p \wedge q) \iff \neg p \vee \neg q$	De Morgan's laws	$\neg(p \vee q) \iff \neg p \wedge \neg q$

Common set equivalences

	Double negation: $\overline{\overline{A}} = A$	
$A \cap U = A$	Identity	$A \cup \varnothing = A$
$A \cap \varnothing = \varnothing$	Domination	$A \cup U = U$
$A \cap A = A$	Idempotence	$A \cup A = A$
$A \cap \neg A = \varnothing$	Negation	$A \cup \neg A = U$
$A \cap (A \cup B) = A$	Absorption	$A \cup (A \cap B) = A$
$A \cap B = B \cap A$	Commutativity	$A \cup B = B \cup A$
$(A \cap B) \cap C = A \cap (B \cap C)$	Associativity	$(A \cup B) \cup C = A \cup (B \cup C)$
$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$	Distribution	$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
$\neg(A \cap B) = \neg A \cup \neg B$	De Morgan's laws	$\neg(A \cup B) = \neg A \cap \neg B$

Don't these look familiar?