Even number of odd vertices

Theorem: $\boxed{\sum_{v \in V} \deg(v) = 2|E|}$ *for every graph* $G = (V, E)$.

Proof:

$\square$

Theorem: *Every graph has an* **even** *number of vertices with* **odd** *degree.*

Proof:

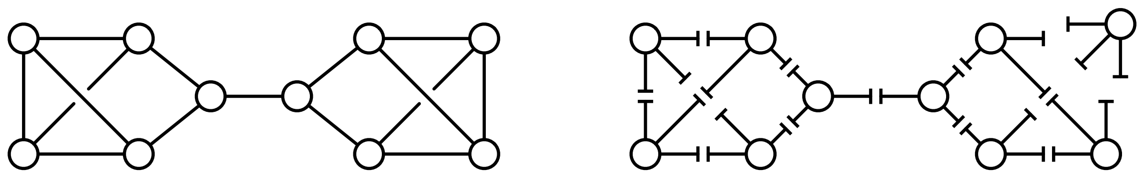
$\square$

Even number of odd vertices

Theorem: $\boxed{\sum_{v\in V} \deg(v) = 2|E|}$ *for every graph* $G = (V, E)$*.*

Proof: Let G be an arbitrary graph.

Split each edge of G into two ‘half-edges’, each with one endpoint.



Any vertex v is incident to $\deg(v)$ half-edges.

Thus, the number of half-edges is $\sum_{v\in V} \deg(v)$.

Every edge was split into exactly two half-edges.

Thus, the number of half-edges is also $2|E|$. $\square$

Theorem: *Every graph has an* **even** *number of vertices with* **odd** *degree.*

Proof: The previous theorem implies that the sum of the degrees is even.

The sum of the even degrees is obviously even.

Thus, the sum of the odd degrees is even. $\square$

Induction on Graphs

Theorem: *$P(G)$ for every graph G.*

Proof: Let $G = (V, E)$ be an arbitrary graph.

Assume that $P(F)$ is true for every proper subgraph F of G.

— Insert proof of $P(G)$ here — **(almost always by cases)**

Thus, by induction, $P(G)$ is true for every graph G. $\square$

Typical induction strategies:

- Let e be an arbitrary edge in G, and let $G' = (V, E \setminus \{e\})$.
- Let v be an arbitrary vertex in G, and let G' be the subgraph of G obtained by deleting v and all its incident edges.
- Let ℓ be an arbitrary leaf (vertex of degree 1) in G, and let G' be the subgraph of G obtained by deleting ℓ and its incident edge.
- Let e be an arbitrary edge, and let G' be the graph obtained by *contracting* e to a single vertex. This approach requires a different inductive hypothesis: Assume $P(F)$ for every *contraction* F of G. See http://en.wikipedia.org/wiki/Edge_contraction.

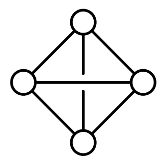
Doug's Induction Trap

Non-Theorem: *For any connected graph G where every vertex has degree 3, it is not possible to disconnect G by removing a single edge.*

"No connected 3-regular graph has a cut edge."

Non-Proof: Every 3-regular graph has an even number of vertices.

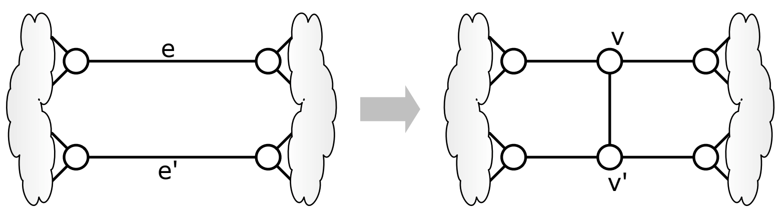
- *Base case:* The clique of size 4 is the smallest connected 3-regular graph. It does not have a cut edge.



- *Induction step:* Let G be an arbitrary 3-regular graph with n vertices, for some $n \geq 4$.

  By the inductive hypothesis, G does not have a cut edge.

  Pick two arbitrary edges in G, split them with two new vertices spanned by a new edge.

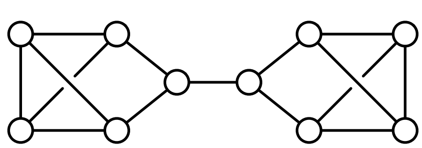


  The new graph G' has $n+2$ vertices.

  Every vertex in G' has degree 3, and G' has no cut edge.

□

Counterexample:



Walk $\implies$ Path

Theorem: *For any graph $G = (V, E)$ and any vertices $u, v \in V$,*
if there is a walk in G from u to v, then there is a path in G from u to v.

Intuition. *If the walk isn't a path, it contains a cycle. Remove it (except for one vertex). Eventually we get a path.*

Proof:

$\square$

Walk $\implies$ Path

Theorem: *For any graph $G = (V, E)$ and any vertices $u, v \in V$, if there is a walk in G from u to v, then there is a path in G from u to v.*

Proof: Let $G = (V, E)$ be an arbitrary graph.

Let $v_0, v_1, \ldots, v_n$ be an arbitrary walk in G.
(We need to prove there is a path in G from v_0 to v_n.)

Assume that for any integer $k < n$, there is a path in G from v_0 to v_k.

There are two cases to consider: $n = 0$ and $n \geq 1$.

- If $n = 0$, then the walk v_0 is a path from v_0 to v_0.
- Suppose $n \geq 1$.

  By the inductive hypothesis, there is a path in G from v_0 to v_{n-1}.

  Either v_n lies on this path or it doesn't.

  - If v_n lies on this path, then stopping at v_n gives us a shorter path in G from v_0 to v_n.
  - If v_n does not lie on this path, then appending v_n gives us a longer path in G from v_0 to v_n.

  In either case, we have found a path in G from v_0 to v_n.

$\square$

Adding or Removing One Edge

Theorem: *Adding an edge from any graph G either joins two components of G or adds a cycle to G, but not both.*

Proof: Let G be an arbitrary graph, let $e = \{u, v\}$ be any edge that is *not* in G, and let $G' = (V, E \cup \{e\})$.

- If u and v are in different components of G, those two components are joined in G'.

  If any cycle in G' contains edge e, then it contains another path from u to v, all of whose edges are in G, which is impossible. Thus, no cycle in G' contains edge e. It follows that every cycle in G' is also a cycle in G.

- If u and v are in the same component of G, then they are connected by a path in G.

  Adding the edge e creates a cycle in G'.

  Consider two arbitrary vertices that are connected by a walk in G'. If the walk contains e, we can use the rest of the new cycle through e instead. Thus, those two vertices are also connected by a walk that does not use e, or in other words, a walk in G.

  Conversely, any walk in G is also a walk in G'.

$\square$

Theorem: *Removing an edge from any graph G either splits a connected component of G or removes a cycle from G, but not both.*

Trees have leaves!

A ***leaf*** is a vertex with degree 1.

Theorem: *Every tree G with more than one vertex has at least two leaves.*

Proof: Let G be an arbitrary connected acyclic graph with more than one vertex.

Because G is connected and has more than one vertex, every vertex has degree at least 1.

Let $v_0, v_1, \ldots, v_n$ be a *maximal* path in G, that is, a path that cannot be made longer by adding a vertex to either end.

Because the path is maximal, it must visit every neighbor of v_n.

If v_n is adjacent to v_i for any $i < n-1$, then $v_i, v_{i+1}, \ldots, v_n, v_i$ is a cycle in G.

Because G is acyclic, no such cycle exists.

Thus, v_n is adjacent to v_{n-1} and nothing else; in other words, v_n is a leaf.

By a similar argument, v_0 is a leaf. $\square$

Trees have leaves!

A ***leaf*** is a vertex with degree 1.

Theorem: *Every tree G with more than one vertex has at least two leaves.*

Proof: Let $G = (V, E)$ be an arbitrary tree.

By definition, G is connected, so every vertex has positive degree.

By earlier results, $\sum_{v \in V} \deg(v) = 2|E| = 2|V| - 2$.

If G has no leaves, then $\deg(v) \geq 2$ for every vertex v, which implies that $\sum_{v \in V} \deg(V) \geq 2|V|$, which is impossible.

Similarly, if G has exactly one leaf, then $\sum_{v \in V} \deg(V) \geq 2|V| - 1$, which is impossible.

We conclude that G has at least two leaves. $\square$

n-node trees have $n-1$ edges

Theorem: *Every tree $G = (V, E)$ has $|V| - 1$ edges.*

Theorem: *Every tree* $G = (V, E)$ *has* $|V| - 1$ *edges.*

Proof: Let G be an arbitrary connected acyclic graph.

By definition of 'acyclic', every subgraph of G is acyclic.

Assume $|E(F)| = |V(F)| - 1$ for every connected proper subgraph F of G.

There are two cases to consider: $|V| = 1$ and $|V| \geq 2$.

- If G has only one vertex, then it has no edges, so $|E| = |V| - 1$.
- Suppose G has more than one vertex.

  Since G is connected, it has at least one edge.

  Let e be an arbitrary edge of G.

  Consider the graph $G \setminus e = (V, E \setminus \{e\})$.

  G is acyclic, so there is no path in $G \setminus e$ between the endpoints of e. Thus, $G \setminus e$ has at least two connected components.

  G is connected, so $G \setminus e$ has at most two connected components.

  Thus, $G \setminus e$ has exactly two connected components. Call them A and B.

  The induction hypothesis implies that $|E(A)| = |V(A)| - 1$.

  The induction hypothesis implies that $|E(B)| = |V(B)| - 1$.

  Because A and B are disjoint, we have $|V| = |V(A)| + |V(B)|$.

  We also have $|E| = |E(A)| + |E(B)| + 1$.

  Simple algebra now implies that $|E| = |V| - 1$.

$\square$

Theorem: *Every tree $G = (V, E)$ has $|V| - 1$ edges.*

Proof: Let G be an arbitrary connected acyclic graph.

By definition of 'acyclic', every subgraph of G is acyclic.

Assume $|E(F)| = |V(F)| - 1$ for every connected proper subgraph F of G.

There are two cases to consider: $|V| = 1$ and $|V| \geq 2$.

- If G has only one vertex, then it has no edges, so $|E| = |V| - 1$.
- Suppose G has more than one vertex.

  Let v be an arbitrary vertex of G, and let $d = \deg(v)$.

  Delete v and all its incident edges from G to get a subgraph G'.

  G' has exactly d connected components.

  For all i, let n_i be the number of vertices in the ith component of G'.

  Then $|V| \;=\; 1 + \sum_{i=1}^{d} n_i$.

  The induction hypothesis implies that for all i, the ith component of G' has $n_i - 1$ edges.

  Thus, $|E| \;=\; d + \sum_{i=1}^{d}(n_i - 1) \;=\; \sum_{i=1}^{d} n_i$.

  Simple algebra now implies that $|E| = |V| - 1$.

$\square$

Theorem: *Every tree $G = (V, E)$ has $|V| - 1$ edges.*

Proof: Let G be an arbitrary connected acyclic graph.

By definition of 'acyclic', every subgraph of G is acyclic.

Assume $|E(F)| = |V(F)| - 1$ for every connected proper subgraph F of G.

There are two cases to consider: $|V| = 1$ and $|V| \geq 2$.

- If G has only one vertex, then it has no edges, so $|E| = |V| - 1$.
- Suppose G has more than one vertex.

  Let v be an arbitrary leaf (vertex of degree 1).

  Delete v and its incident edge from G to get a subgraph G'.

  For any vertices u and w in G', there is a *path* from u to w in G. This path cannot pass through v (because $\deg(v) = 1$), so it is also a path in G'. Thus, G' is connected.

  G' has $|V| - 1$ vertices.

  Thus, the inductive hypothesis implies that G' has $|E| - 2$ edges.

  Simple algebra now implies that $|E| = |V| - 1$.

$\square$