EECE 320: Discrete Structures & Algorithms
Graphs and Trees

Some examples with complete solutions

1. **Euler's Theorem states that for any planar graph $G = (V, E)$ with n vertices, e edges and f faces, $n - e + f = 2$. Prove using the notion of a dual graph.**

Proof: Let G^* be the dual of a graph G. G has f faces and its dual has a vertex for each face in G therefore the number of vertices in G^* is f.

Let T be some spanning tree for G. $G - T$ is the graph with the same vertex set as G but with only those edges that are in G but not in the spanning tree T. Every edge in G lies at the boundary of two faces; hence, every edge in $G - T$ is at the boundary of two faces. Consider the graph T^* that includes those edges in G^* that correspond to those edges in $G - T$. In other words, if e is an edge in $G - T$ and e separates faces F_i and F_j in G, then T^* includes the edge in G^* that connects the vertices in G^* corresponding to faces F_i and F_j.

Since T is a tree, it contains no cycles; therefore there is an edge in $G - T$ corresponding to each face in G. Thus T^* has an edge for each vertex in G^*. Then:

(a) <u>T^* does not contain a cycle.</u> Why? Given any cycle in T^* we can draw a cycle on the faces of G that participate in the cycle. This cycle must partition the vertices of G into two non-empty sets and only crosses the edges in $G - T$. This implies that T has more than one connected component, which is possible only if T is not a spanning tree. This contradicts the fact that T is spanning and therefore T^* contains no cycles.

(b) <u>T^* is a spanning tree.</u> Why? Suppose T^* does not connect all the faces. Let F_i and F_j be two faces that are not connected in T^*. This implies that T must include a cycle separating F_i from F_j, and therefore cannot be a tree. This contradicts the fact that T is a spanning tree.

Because T^* is a spanning tree, we know that $f = e_{T^*} + 1$ (f is the number of vertices in the tree T^* and e_{T^*} is the number of edges in T^*). Further, we know that $n = e_T + 1$ because T is a spanning tree for g and e_T is the number of edges in the spanning tree T.

Combining these two equations:

$$n + f = e_T + e_{T^*} + 2.$$

But $e_T + e_{T^*} = e$ because T^* contains an edge for every edge in $G - T$. We conclude that

$$n - e + f = 2,$$

which completes the proof of Euler's Theorem. □

2. **If G is a simple planar graph with $n > 2$ vertices then prove that (a) G has a vertex of degree at most 5, and (b) G has at most $3n - 6$ edges.**

   Solution. This question will rely on Euler's Theorem. It is also useful to remember that a simple graph is a graph where there is at most one edge between every pair of vertices.

   Proof: In any graph, the degree sum is $2e$. If we suppose that all vertices have degree ≥ 6 then the degree sum is at least $6n$. As a result, we must have

   $$6n \leq 2e \implies 3n \leq e. \tag{1}$$

   Since G is a simple graph, every face must have at least three edges (the smallest closed face is a triangle). Counting the number of edges in G by counting the number of edges per face, and noting that each edge belongs to two faces, we can obtain the inequality

   $$3f/2 \leq e \implies 3f \leq 2e. \tag{2}$$

   Using Euler's Theorem, we know that $n - e + f = 2$ or that

   $$3n - 3e + 3f = 6. \tag{3}$$

   Using (1) and (2) in (3), we obtain the inequality

   $$e - 3e + 2e \geq 6 \implies 0 \geq 6,$$

   which is clearly false. Therefore it is not possible that all vertices of G have degree ≥ 6.

   From (3), we also infer that $3n - 6 = 3e - 3f$. Using (2), we can write the following equality: $3f = 2e - \delta$ for some $\delta \geq 0$. That allows us to conclude that $3n - 6 = e + \delta \implies e = 3n - 6 - \delta \implies e \leq 3n - 6$. □

3. **Prove that every planar graph G can be 6-coloured.**

Proof: We will consider a proof by induction.

Base case: If G consists of 6 edges or fewer then the result holds trivially.

Induction step: Let us suppose that we can colour a planar graph with n vertices using six colours. We need to show that a planar graph with $n + 1$ vertices can be coloured with six colours. Every planar graph has at least one vertex with degree at most 5. Let v be a vertex in G with degree at most 5. Let $G - \{v\}$ be the graph obtained by deleting the vertex v and the edges incident on v. Now, $G - \{v\}$ is a planar graph with n vertices and has a 6-colouring (by the induction hypothesis). Consider some colouring of $G - \{v\}$ with at most 6 colours. If we include v and the edges associated with v to recreate G then v has at most 5 neighbours; v's neighbours use at most 5 colours so we can always colour v using the remaining colour to obtain a valid 6-colouring of G. □

4. **A graph G is 2-colourable iff it contains no odd length cycle.**

Proof: (a) G is 2-colourable $\implies$ G contains no odd length cycle. Assume that G is 2-colourable and consider some 2-colouring of G. Consider an arbitrary cycle of successive vertices $v_1, v_2, \ldots, v_k, v_1$. The vertices v_i must be one colour for all even i and the other colour for odd i. Since v_1 and v_k must have different colours, k must be even. Thus any cycle must be of even length.

(b) G contains no odd length cycle $\implies$ G is 2-colourable.

Let us consider a spanning tree T of G. A 2-colouring of a tree can be defined by selecting any fixed vertex v, and colouring a vertex one colour if the (unique) path to it from v has odd length, and colouring it with the other colour if the path has even length.

To verify that adjacent vertices in the tree get different colours, let $e ::= xy$ be an edge in the tree. There is a unique path from v to x. If this path traverses e, it must consist of a path from v to y followed by the e traversal to x. If this path does not traverse e, then it can be extended to a path to y by adding a final traversal of e. In either case, the paths to these vertices from v differ by a single traversal of e, and so the lengths of the paths differ by 1; in particular, one is of odd length and the other is of even length, so x and y are differently-coloured.

Let xy be an edge not in T, and consider the unique paths from v to x and from v to y in T. Exactly one of these two paths must have odd length; otherwise, these two paths together with the edge xy would form an odd length cycle. But this means x and y are coloured differently.

$\square$

5. **(The Marriage Theorem) Let G be a bipartite graph with bipartition X and Y. Then there is a maximal matching from X to Y if and only if Hall's condition is satisfied: $|N(A)| \geq |A|$ for all subsets A of X. Here $N(A)$ denotes the set of neighbours of the vertices in A.**

Proof: The necessity is obvious. The sufficient part can be shown using induction. For simplicity of exposition we will assume that X is a set of n boys and Y is a set of n girls.

The case of $n = 1$ and a single pair liking each other requires a mere technicality to arrange a match. Assume we have already established the theorem for all cases when $|X| = k$ and $|Y| = k$ with $k < n$. For the case of n girls and boys, the marriage condition may be satisfied with room to spare (each of the inequalities is strict) or just barely.

In the first case, there is enough room for the first girl to marry whomever she likes; Hall's condition will still be satisfied for the remaining $(n-1)$ girls and $(n-1)$ boys. Indeed, every $0 < r < n$ girls like more than r boys. One of those boys may have been the one who married the first girl - but without whom there are still at least r boys. So, after marrying off any eligible pair we shall be left with $(n-1)$ girls and boys for whom the marriage condition still holds and, by the inductive hypothesis, a complete match is possible.

In the second case, there are $r < n$ girls who like exactly r boys. By the inductive hypothesis, a complete match exists for these r girls so they can be married to the r boys they like. Consider any s of the remaining $n - r$ girls. The r married girls plus these s girls must like at least $r + s$ boys as assured by Hall's condition. Since the r married girls do not like boys other than the r they married, the s girls must like s boys other than the married boys. Hence the remaining $n - r$ girls satisfy the marriage condition with the unmarried boys; and so a complete match is possible for the remaining girls with the remaining boys, providing a complete match for all the girls. $\square$

6. **An *Eüler path* is a path that uses every edge of a graph, and uses each edge exactly once. An *Eüler circuit* is an Eüler path that is a circuit, i.e., the starting and ending vertex are the same. Prove that a connected graph has an Eüler path (that is not a circuit) if and only if it has exactly two vertices with odd degree. Also prove that a connected graph has an Eüler circuit if and only if all vertices have even degree.**

   We will prove that the conditions suffice for the existence of an Eüler path and an Eüler circuit respectively. It is easy to see that they are necessary (left as an exercise).

   Proof: We shall use prove the result using induction on $|E|$, the number of edges in the graph.

   Base case. When $|E| = 1$, it is trivial to see that the statement is true (of course, we do not have Eüler circuit in this case).

   Induction step. Now assume the statement is true for $|E| \leq n$. In other words, when $|E| \leq n$, if G has exactly two vertices with odd degree, then G has an Eüler circuit; If all vertices of G have even degree then it has an Eüler circuit. We shall show that the above statement is also valid for $|E| = n + 1$.

   First consider the case that all vertices of G have even degree. Select a vertex x and edge e that connects x with y. If we delete the edge e, we get a new graph $\hat{G}$. We claim that the new graph has to be connected. If not, then x and y will be in two separated graphs respectively. Say x is in a connected graph $\tilde{G}$. Note that x is the only vertex with odd degree in $\tilde{G}$. Thus the sum of degrees of all vertices in $\tilde{G}$ is an odd number. This contradicts that fact that the degree sum of a graph is always even. Thus $\hat{G}$ has to be connected. Since $\hat{G}$ is connected, $|\hat{E}| = n$, and has exactly two vertices x and y with odd degree, by induction, there exists an Eüler path P in $\hat{G}$, which must have x and y as ending vertices. Now add the edge from y to x to the path P and we obtain an Eüler circuit in G.

   Now consider the case that G has exactly two vertices, say x and y, with odd degree. First assume that x and y are connected by an edge e. After deleting e, we get a new graph $\hat{G}$ with $|\hat{E}| = n$ and all vertices have even degree. But $\hat{G}$ could be either connected or disconnected. If $\hat{G}$ is connected, then we have an Eüler circuit in $\hat{G}$. Adding the edge e to the Eüler circuit, we get an Eüler path in G from x to y. If $\hat{G}$ has two components, say G_1 and G_2 then all vertices in both G_1 and G_2 have even degree and $|E_i| \leq n, i = 1, 2$. As a result, there exist Eüler circuits C_i in G_i respectively. Now adding the edge e to C_1 and C_2, we get an Eüler path in G from x to y.

   If there is no edge connecting x and y, then consider an edge e that connects x to another vertex z. Deleting e, we get a new graph $\hat{G}$ that has exactly two vertices, z and y, of odd degree. If $\hat{G}$ is connected, then by induction, there exists an Eüler path in $\hat{G}$ connecting y and z. By adding e, we get an Eüler path from y to x. But $\hat{G}$ could be disconnected. In such a case, x and z are necessarily in different components G_1 and G_2, respectively (otherwise, $\hat{G}$ will be connected). y can only be in the component G_2 with z else we would have a contradiction to the fact that the degree sum is even. Now all vertices of G_1 have even degree and G_2 has exactly two vertices, y and z, with odd degree. By induction, we have an Eüler circuit C_1 in G_1 and an Eüler path C_2 in G_2 from z to y. Now connect C_1 and C_2 by e: we get an Eüler path from x to y. $\square$