A ***binary relation*** between A and B is a subset of $A \times B$.

A is called the ***domain***; B is called the ***codomain***.

- $A =$ *Students*; $B =$ *Classes*; $R = \{(a, b) \mid a$ is taking b this semester$\}$.
- $A =$ *People*; $B =$ *People*; $R = \{(a, b) \mid a$ is b's ancestor$\}$
- *has downloaded* $\subseteq$ *People* $\times$ *Songs*
- *likes* $\subseteq$ *People* $\times$ *IceCreamFlavors*
- *acted in a movie with* $\subseteq$ *Actors*^{2}
- *is a multiple of* $\subseteq \mathbb{Z}^2$
- $\leq \subseteq \mathbb{R}^2$
- $\in \subseteq \mathbb{Z} \times 2^{\mathbb{Z}}$

Equivalently, a binary relation is a predicate with two free variables.

$\mathit{IsTaking}(x, y)$	$\mathit{Ancestor}(a, b)$	$a \leq b$	$a = b^3$
$x = y$	k divides n	$x^2 + y$ is a multiple of 17	
$A \subset B$	$\lvert A \rvert < \lvert B \rvert$	$A \cap B = \varnothing$	$x \in A$
$p \rightarrow q$	$p \wedge q$	$W \equiv X$	$p = \mathit{IsAlive}(x)$

$(\mathit{BeatsAtChess}(x, \mathit{Kasparov}) \wedge \neg \mathit{SmellsLike}(y, \mathit{Roses})) \vee \mathit{Eats}(x, y)$

A relation between A and B is a ***function*** if and only if each element of A is related to *exactly one* element of B.

A ***function*** f from A to B is an assignment of *exactly* one element $f(a) \in B$ to each element $a \in A$. This is written $\boldsymbol{f\colon A \to B}$.

- A is called the ***domain*** of f.
- B is the ***codomain*** of f.
- $f(a)$ is called the ***image*** of a.

For example:

- $A =$ brands of beer
- $B = \{x/4 \mid x \in \mathbb{N}\}$
- f maps beer brands to prices in dollars
- $f($*La Fin Du Monde*$) = 5$

Small Brewery Bottled Beer

Anchor Steam - San Francisco, CA	4.25
Bavik Premium Pils - Belgium	4.50
Blue Moon - white ale, Colorado	3.50
Carlsberg Beer - lager from Denmark	4.50
Chimay Blue Cap - Belgium; 18 proof	7.50
Delirium Tremens - Belgian ale; 17 proof	8.00
Founders Red's Rye - red ale brewed w/rye	4.50
Founders DIRTY BASTARD - Scottish ale brewed with ten varieties of imported malts, a bold and powerful ale	$4.50
Goose Island - India Pale Ale, A hop lover's dream!	$4.50
Grolsch Lager - swingtop bottle	5.00
Konig Ludwig - Royal Bavarian Hefe-Weizen	4.50
La Fin Du Monde - Belgian style golden ale	5.00
Harp Lager - Ireland	4.25
Hoegaarden - Belgium, sweet and sour, gently bitter	5.00
HofBrau - original, from Germany	4.50
Left Hand JUJU Ginger - pale ale brewed with ginger, from Colorado	$4.50
Lindeman's Framboise - lambic style ale with real raspberries added!	$8.50
Newcastle Brown Ale - England	4.25
Pilsner Urquell - Czech Republic	4.50
Paulaner Thomas Brau - German N/A lager	3.50
Samuel Adams - lager from Boston	4.00
Sterken's White Ale - Belgian White Ale	4.50
Sierra Nevada Pale Ale - from CA	4.00
Staropramen - pilsner from Prague	4.25
Strongbow - dry English cider	4.25
Steinlager - lager from New Zealand	4.50
Theakston Old Peculiar - English dark brown complex ale	5.00
Tucher Light Hefe Weizen - Germany	5.00
Warsteiner Dunkel - German dark	4.25
Wolaver's Brown Ale - organic!	4.50
Wolaver's IPA - organic!	4.50
Xingu - black beer from Brazil	4.75

- The ***range*** of f is the set of images of elements of A:

$$\text{range}(f) = \{f(a) \mid a \in A\} = \{b \in B \mid \exists a \in A\colon f(a) = b\}$$

- The ***image*** of a subset $S \subset A$ is

$$\boxed{f(S) = \{f(a) \mid a \in S\}}$$

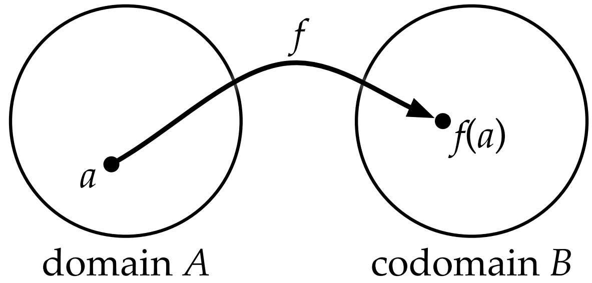
  In particular, $\text{range}(f) = f(A)$.

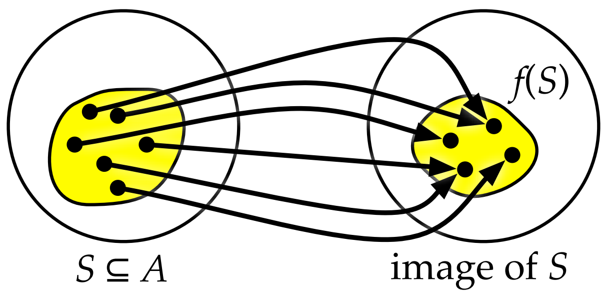
- The ***preimage*** of an element $b \in B$ is

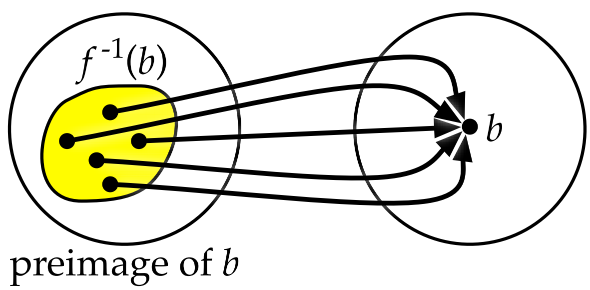
$$\boxed{f^{-1}(b) = \{a \mid f(a) = b\}}$$

- The ***preimage*** of a subset $T \subseteq B$ is

$$\boxed{f^{-1}(T) = \{a \mid f(a) \in S\}}$$







Two things to prove

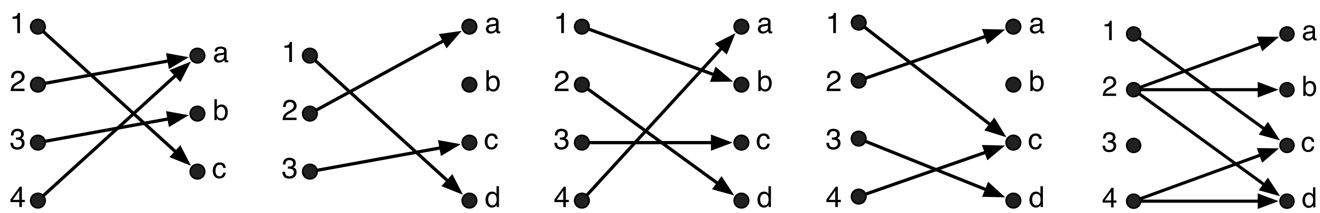
- For any subset $T \subseteq B$, we have $f(f^{-1}(T)) = T$.
- For any subset $S \subseteq A$, we have $f^{-1}(f(S)) \supseteq S$.

Kinds of functions

- A function $f\colon A \to B$ is ***one-to-one*** (or an ***injection***) if every element of B has *at most* one preimage.

- A function $f\colon A \to B$ is ***onto*** (or a ***surjection***) if every element of B has *at least* one preimage, or in other words, if $f(A) = B$.

- A function $f\colon A \to B$ is a ***bijection*** if every element of B has *exactly* one preimage, or in other words, if f is both one-to-one and onto.



onto | one-to-one | bijection | neither | not a function

Examples (all functions $f\colon \mathbb{Z} \to \mathbb{Z}$):

- $f(x) = x + 1$

- $f(x) = 2x$

- $f(x) = \lfloor x/2 \rfloor$

  $\lfloor\ \rfloor$ is the ***floor*** function: round down to the nearest integer.

  $$\lfloor 2 \rfloor = 2 \qquad \lfloor \pi \rfloor = 3 \qquad \lfloor -\pi \rfloor = -4$$

- $f(x) = 42$

- $f(x) = \begin{cases} x + 1 & \text{if } x \text{ is even} \\ x - 1 & \text{if } x \text{ is odd} \end{cases}$

Kinds of functions

More Examples:

- $f : \mathbb{N} \to \mathbb{N}$ where $f(x) = x^2$ is one-to-one but not onto.
- $f : \mathbb{Z} \to \mathbb{Z}$ where $f(x) = x^2$ is neither one-to-one nor onto.
- $f : \mathbb{R} \to \mathbb{R}_{\geq 0}$ where $f(x) = x^2$ is onto but not one-to-one.
- $f : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ where $f(x) = x^2$ is both one-to-one and onto.