Fourier series representation of periodic signals – Chapter 3[2]

We have seen that a signal can generally be represented as a linear combination of shifted impulse (or sample) functions.

We will show that a signal can also be represented as a linear combination of complex exponential functions, provided certain conditions are satisfied.

Why is this useful? Section 3.2

Reason is due to this important fact: The output of an LTI system due to a complex exponential input is the same complex exponential multiplied by a (possibly complex) gain factor.

We say that complex exponentials are

[2]These slides are based on lecture notes of Professors L. Lampe, C. Leung, and R. Schober

eigenfunctions of LTI systems; the gain factors are termed *eigenvalues*. (cf. eigenvectors and eigenvalues of a matrix)

We now prove the following.

Continuous-time: If $x(t) = e^{st}$ is input to a LTI system with impulse response $h(t)$, the output $y(t)$ is

$$y(t) = H(s)e^{st}$$

where

$$H(s) = \int_{-\infty}^{\infty} h(\tau)\, e^{-s\tau}\, d\tau$$

Proof:

Example:

Suppose $y(t) = x(t-1)$ and $x(t) = e^{j2\pi t}$.

Example:

Suppose $y(t) = x(t-1)$ and $x(t) = \cos 2\pi t + \cos 3\pi t$.

Discrete-time: If $x[n] = z^n$ is input to a LTI system with impulse response $h[n]$, the output $y[n]$ is

$$y[n] = H(z)z^n$$

where

$$H(z) = \sum_{k=-\infty}^{\infty} h[k]\ z^{-k}$$

Proof:

Fourier series representation of CT periodic signals – Section 3.3

Let $x(t)$ be a periodic signal with fundamental period T. Then if certain conditions (Dirichlet, pp. 197-200) are satisfied, we can represent $x(t)$ as a Fourier series (FS) (an infinite sum of complex exponentials):

$$\boxed{x(t) = \sum_{k=-\infty}^{\infty} a_k \, e^{jk\left(\frac{2\pi}{T}\right)t}}$$

where the (possibly complex) Fourier coefficients $\{a_k\}$ are given by

$$a_k = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x(t) \, e^{-jk\left(\frac{2\pi}{T}\right)t} \, dt,$$

$$k = 0, \pm 1, \pm 2, \ldots$$

The above equations are referred to as the *synthesis* and *analysis* equations respectively.

Section 3.4 provides a good discussion of the *convergence* of the FS representation.

Notes:

1. Using Euler's relationship, we can also express the FS representation as an infinite sum of sine and cosine terms.

2. Given $x(t)$, we can determine $\{a_k\}_{k=-\infty}^{\infty}$; conversely, we can reconstruct $x(t)$ from $\{a_k\}_{k=-\infty}^{\infty}$.

   $\{a_k\}_{k=-\infty}^{\infty}$ give the *frequency-domain* description of the signal and are called its *spectral coefficients*.

3. A periodic signal $x(t)$ of fundamental period T has components at frequencies $0, \pm\frac{2\pi}{T}, \pm\frac{4\pi}{T}, \ldots$, i.e. at multiples of the fundamental frequency $\omega_0 = \frac{2\pi}{T}$ or $f_0 = \frac{1}{T}$.

The component at freq nf_0 is called the *nth harmonic*.

4. *Conjugate symmetry property:* If $x(t)$ is real, then $a_{-k} = a_k^*$.

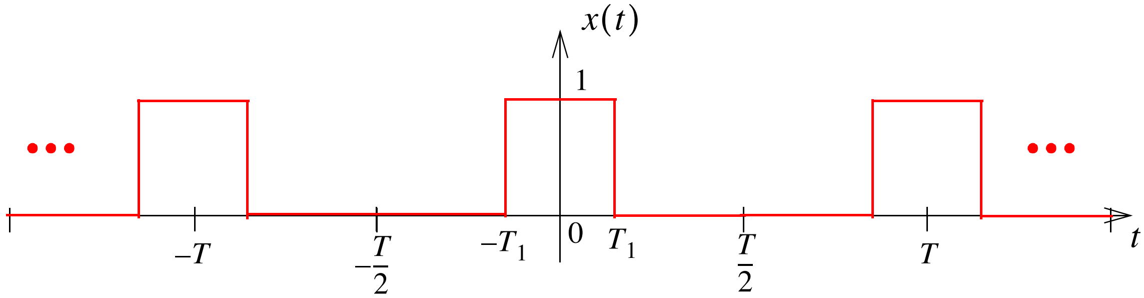
Proof:

As a result,

$$|a_{-k}| = |a_k^*| \;=\; |a_k|$$
$$\text{and} \quad \angle a_{-k} = \angle a_k^* \;=\; -\angle a_k \;.$$

Thus, when $x(t)$ is real, the *amplitudes* of the spectral coefficients have *even symmetry* whereas their *phases* have *odd symmetry*.

Example: Periodic square wave shown below.



Periodic square wave

We would like to determine its Fourier or spectral coefficients $\{a_k\}_{k=-\infty}^{\infty}$.

Recall that

$$a_k = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)\; e^{-jk\left(\frac{2\pi}{T}\right)t}\; dt, k = 0, \pm 1, \pm 2, \ldots$$

In this case, we have

$$\begin{aligned}
a_k &= \frac{1}{T}\int_{-T_1}^{T_1} e^{-jk\left(\frac{2\pi}{T}\right)t}\; dt \\
&= -\frac{1}{jk2\pi}\; e^{-jk\left(\frac{2\pi}{T}\right)t}\Big]_{-T_1}^{T_1} \\
&= \frac{2}{k2\pi}\underbrace{\left[\frac{e^{jk\left(\frac{2\pi}{T}\right)T_1} - e^{-jk\left(\frac{2\pi}{T}\right)T_1}}{2j}\right]}_{\sin k\left(\frac{2\pi}{T}\right)T_1} \\
&= \frac{\sin k\left(\frac{2\pi}{T}\right)T_1}{k\pi} \\
&= \frac{\sin k\left(\frac{2\pi}{T}\right)T_1}{k\pi\frac{2T_1}{T}} \times \frac{2T_1}{T} \\
&= \frac{2T_1}{T}\,\mathrm{sinc}\left(\frac{2kT_1}{T}\right)
\end{aligned}$$

As expected, the amplitudes of the FS coefficients have even symmetry; in this case, the phases are 0.

e.g. $T_1 = T/4$

Then, the average value is

$$a_0 = \frac{1}{2}$$

and

$$\begin{aligned} a_k &= \frac{1}{2}\,\mathrm{sinc}\left(\frac{k}{2}\right) \\ &= \frac{1}{2}\,\frac{\sin \pi k/2}{\pi k/2} \\ &= \frac{\sin \pi k/2}{\pi k}\,. \end{aligned}$$

k	1	2	3	4	5	6	7
a_k	$\frac{1}{\pi}$	0	$-\frac{1}{3\pi}$	0	$\frac{1}{5\pi}$	0	$-\frac{1}{7\pi}$

Properties of CT Fourier series – Section 3.5

See Table 3.1, p. 206 for a list of CTFS properties. We will look at a few of the more commonly used ones here. Most of these properties can also be obtained from our future study of CT Fourier transform.

Notation: The pairing of a periodic signal $x(t)$ and its FS coefficients a_k is represented by

$$x(t) \stackrel{\mathcal{FS}}{\longleftrightarrow} a_k$$

1. *Linearity – Section 3.5.1*
   Let $x(t)$ and $y(t)$ be two periodic signal with period T and $x(t) \stackrel{\mathcal{FS}}{\longleftrightarrow} a_k$, $y(t) \stackrel{\mathcal{FS}}{\longleftrightarrow} b_k$.
   Then,

$$z(t) = Ax(t) + By(t) \stackrel{\mathcal{FS}}{\longleftrightarrow} c_k = Aa_k + Bb_k \ .$$

Proof:

2. *Time shift – Section 3.5.2*

   If $x(t) \overset{\mathcal{FS}}{\longleftrightarrow} a_k$, then

   $x(t-t_0) \overset{\mathcal{FS}}{\longleftrightarrow} e^{-jk\left(\frac{2\pi}{T}\right)t_0} a_k$.

   As a result, when a periodic signal is shifted in time, the magnitudes of its FS coefficients are unchanged.

   Proof:

3. *Time reversal – Section 3.5.3*
If $x(t) \stackrel{\mathcal{FS}}{\longleftrightarrow} a_k$, then $x(-t) \stackrel{\mathcal{FS}}{\longleftrightarrow} a_{-k}$.

As a result, if $x(t)$ is even, i.e. $x(-t) = x(t)$, then $a_{-k} = a_k$, i.e. the FS coefficients are also even.

Moreover, if $x(t)$ is odd, i.e. $x(-t) = -x(t)$, then $a_{-k} = -a_k$, i.e. the FS coefficients are also odd.

Proof:

4. *Time Scaling – Section 3.5.4*
If $x(t) = \sum_{k=-\infty}^{\infty} a_k \; e^{jk\left(\frac{2\pi}{T}\right)t}$,
then $x(\alpha t) = \sum_{k=-\infty}^{\infty} a_k \; e^{jk\left(\alpha\frac{2\pi}{T}\right)t}$.

Note that each of the “tones” in $x(t)$ are simply compressed in time by a factor of α.

Proof:

5. *Multiplication – Section 3.5.5*

If $x(t)$ and $y(t)$ are periodic with a common period T and FS coefficients $\{a_k\}$ and $\{b_k\}$ respectively, then

$$x(t)y(t) \overset{\mathcal{FS}}{\longleftrightarrow} h_k = \sum_{l=-\infty}^{\infty} a_l b_{k-l}.$$

Note that the RHS is the DT convolution of the sequences representing the FS coefficients of $x(t)$ and $y(t)$.

Proof:

6. *Conjugation – Section 3.5.6*

If $x(t) \overset{\mathcal{FS}}{\longleftrightarrow} a_k$, then $x^*(t) \overset{\mathcal{FS}}{\longleftrightarrow} a^*_{-k}$.

The conjugate symmetry property of the CTFS we saw previously follows easily from the above conjugation property.

Proof:

7. *Parseval's relation – Section 3.5.7*

If $x(t) \stackrel{\mathcal{FS}}{\longleftrightarrow} a_k$, then
$\frac{1}{T}\int_T |x(t)|^2 \ dt = \sum_{k=-\infty}^{\infty} |a_k|^2$.

The LHS is the average power of $x(t)$. Also since

$$\frac{1}{T}\int_T \left|a_k \ e^{jk\left(\frac{2\pi}{T}\right)t}\right|^2 \ dt = \frac{1}{T}\int_T |a_k|^2 \ dt = |a_k|^2,$$

we see that $|a_k|^2$ is the average power in the kth harmonic component of $x(t)$.

Parseval's relation states that the average power in $x(t)$ is equal to the sum of the average powers in all its harmonics.

Proof: Problem 3.46.

Fourier series representation of DT periodic signals – Section 3.6

Let $x[n]$ be a periodic signal with fundamental period N. Then, as it is shown in the textbook, equations (3.94) and (3.95), we have the following DT Fourier series (DTFS) pair

$$x[n] = \sum_{k=<N>} a_k \, e^{jk\left(\frac{2\pi}{N}\right)n}$$

where the (possibly complex) Fourier coefficients $\{a_k\}$, also known as spectral coefficients, are given by

$$a_k = \frac{1}{N} \sum_{n=<N>} x[n] \, e^{-jk\left(\frac{2\pi}{N}\right)n}$$

The notation $\sum_{k=<N>}$ is used to indicate that the summation is over N consecutive integers, starting with any value of k.

The above two equations are referred to as the *synthesis* and *analysis* equations respectively.

There are similarities to as well as differences from the CT case.

Remarks:

1. The synthesis equation in the *DT case involves a finite sum*, in sharp contrast to the *CT case which involves an infinite sum*. Hence, unlike the CT case, there are no convergence issues in the DT case.

2. Unlike the CT case, $a_k = a_{k+N}$.

3. An important set of DT complex exponentials is defined as

$$\phi_k[n] = e^{jk\left(\frac{2\pi}{N}\right)n}, k = 0, \pm 1, \pm 2, \ldots$$

Note: $\phi_k[n] = \phi_{k+iN}[n]$.

4. *Fact:*

$$\sum_{n=<N>} \phi_k[n] = \sum_{n=<N>} e^{jk\left(\frac{2\pi}{N}\right)n}$$

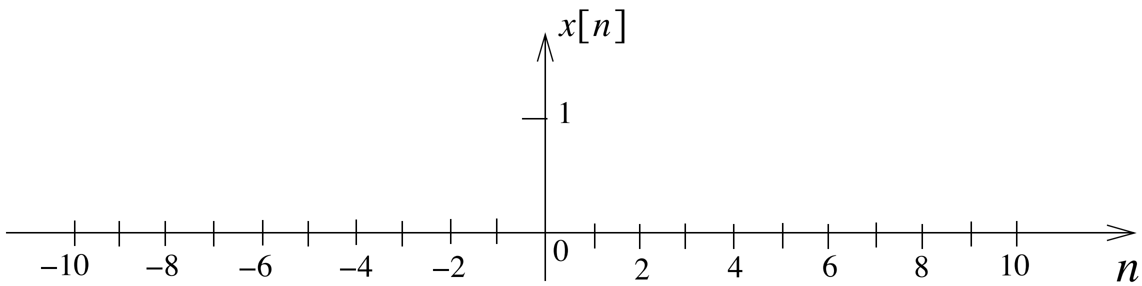
$$= \begin{cases} N, & k = 0, \pm N, \pm 2N, \ldots \\ 0, & \text{otherwise.} \end{cases}$$

Proof:

Example:

Suppose $x[n] = \sin\left(\frac{2\pi}{5}\right)n$, a signal with fundamental period 5.

n	0	1	2	3	4	5	6
$x[n]$	0	0.95	0.59	-0.59	-0.95	0	0.95



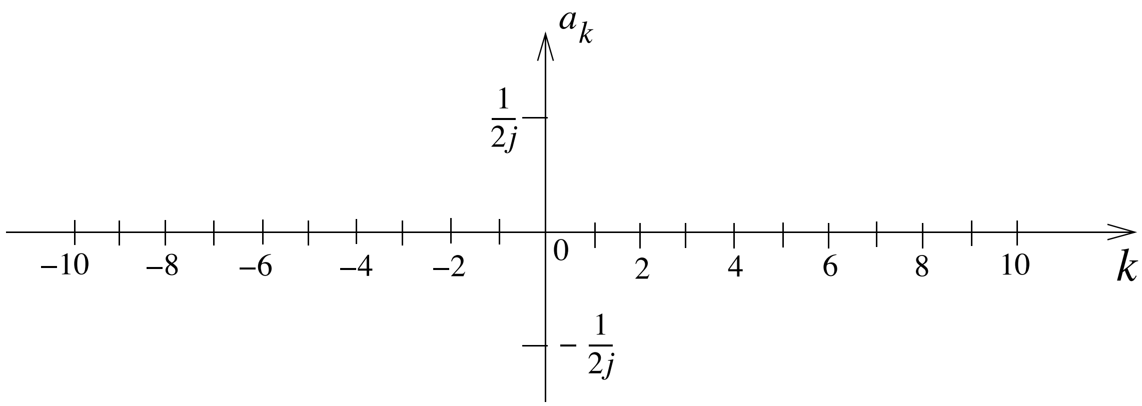
$\sin\left(\frac{2\pi}{5}\right)n$

Then, we can write

$$x[n] = \frac{1}{2j}\left[e^{j\left(\frac{2\pi}{5}\right)n} - e^{-j\left(\frac{2\pi}{5}\right)n}\right].$$

By direct comparison with the *synthesis equation for the DTFS*, we obtain

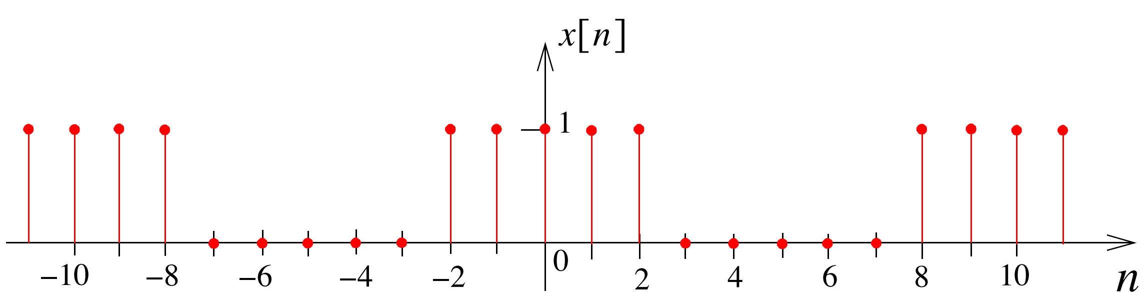
$$a_1 = \frac{1}{2j}, \quad a_{-1} = -\frac{1}{2j}\,.$$



Spectral coefficients of $\sin\left(\frac{2\pi}{5}\right) n$

Example: Consider the following signal with fundamental period $N = 10$: in the interval $-5 \leq n \leq 5$,

$$x[n] = \begin{cases} 1, & -2 \leq n \leq 2 \\ 0, & \text{otherwise.} \end{cases}$$

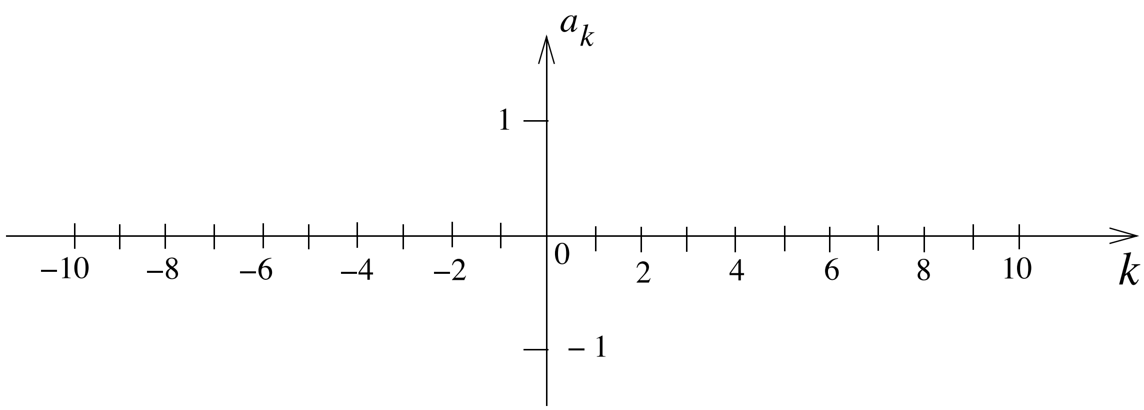


DT periodic square wave

Then,

$$a_k = \frac{1}{10} \sum_{n=-2}^{2} e^{-jk\left(\frac{2\pi}{10}\right)n}$$

n	-5	-4	-3	-2	-1	0	1	2	3	4
a_n	0.1	0.0	-0.12	0.0	0.32	0.5	0.32	0.0	-0.12	0.0



FS coefficients of DT periodic square wave

More generally, suppose that $x[n]$ is periodic with fundamental period N and in the interval $-\left\lfloor \frac{N}{2} \right\rfloor \leq n \leq \left\lfloor \frac{N-1}{2} \right\rfloor$,

$$x[n] = \begin{cases} 1, & -N_1 \leq n \leq N_1 \\ 0, & \text{otherwise.} \end{cases}$$

Then, from Example 3.12 on p. 218, we have

$$a_k = \begin{cases} \frac{\sin\left[2\pi k\left(N_1+\frac{1}{2}\right)/N\right]}{N\sin(\pi k/N)}, & k \text{ not a multiple of } N \\ \frac{2N_1+1}{N}, & k \text{ a multiple of } N. \end{cases}$$

Properties of DT Fourier series – Section 3.7

Although there are *important differences* such as the *periodicity of the FS coefficients* and the *Gibbs phenomenon* between the CT and DT Fourier series, there are also *many similarities*.

See Table 3.2 on p. 221 for a list of DTFS properties. Most of them can be derived in similar fashion to the CTFS cases.

We next look at a couple of properties to illustrate the similarities and differences.

Notation: The pairing of a periodic signal $x[n]$ and its FS coefficients a_k is represented by

$$x[n] \overset{\mathcal{FS}}{\longleftrightarrow} a_k$$

1. *Multiplication – Section 3.7.1*

   If $x[n]$ and $y[n]$ are periodic with a common period N and FS coefficients $\{a_k\}$ and $\{b_k\}$ respectively, then

   $$x[n]y[n] \stackrel{\mathcal{FS}}{\longleftrightarrow} d_k = \sum_{l=<N>} a_l b_{k-l}.$$

   Note that the RHS is called the DT *periodic convolution* of the sequences representing the FS coefficients of $x[n]$ and $y[n]$.

2. *Parseval's relation – Section 3.7.3*

   If $x[n] \stackrel{\mathcal{FS}}{\longleftrightarrow} a_k$, then

   $$\frac{1}{N}\sum_{n=<N>} |x[n]|^2 = \sum_{k=<N>} |a_k|^2.$$

   The LHS is the average power of $x[n]$. Also since

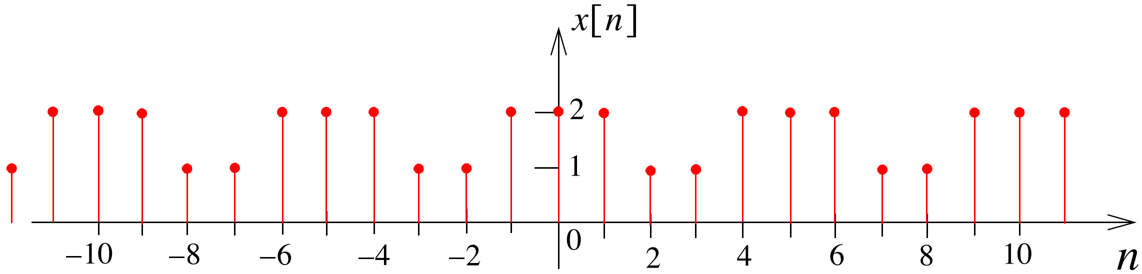
$$\frac{1}{N}\sum_{n=<N>} \left|a_k\, e^{jk\left(\frac{2\pi}{N}\right)n}\right|^2 = \frac{1}{N}\sum_{n=<N>} |a_k|^2$$
$$= |a_k|^2,$$

we see that $|a_k|^2$ is the average power in the kth harmonic component of $x[n]$.

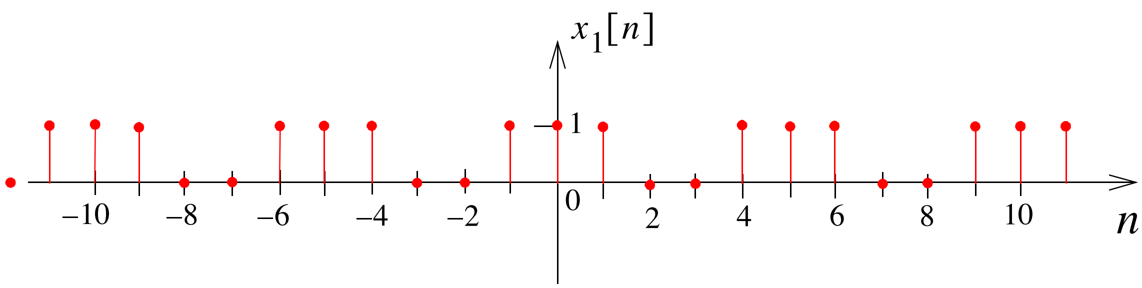
Parseval's relation states that the average power in $x[n]$ is equal to the sum of the average powers in all its harmonics.

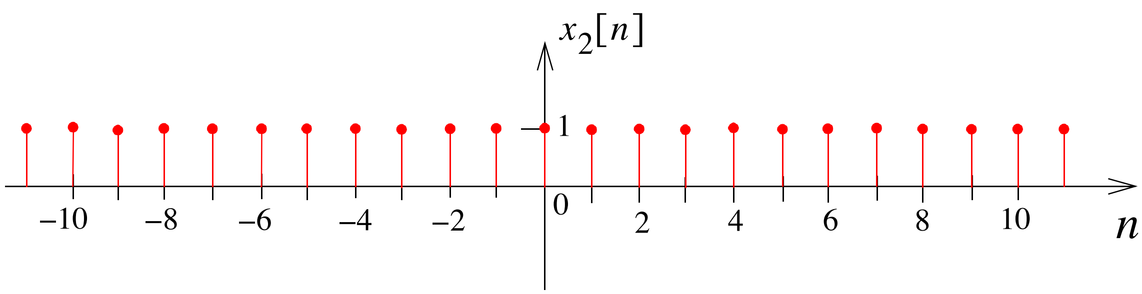
Example:

Determine the DTFS coefficients of the signal $x[n]$, with a fundamental period of 5, as shown in the following figure.



To solve this problem, we can make use of the linearity property of the DTFS by noting that $x[n] = x_1[n] + x_2[n]$, where $x_1[n]$ and $x_2[n]$ are shown in the following figures.





If

$$x[n] \overset{\mathcal{FS}}{\longleftrightarrow} a_k, x_1[n] \overset{\mathcal{FS}}{\longleftrightarrow} b_k, x_2[n] \overset{\mathcal{FS}}{\longleftrightarrow} c_k$$

then

$$a_k = b_k + c_k \ .$$

Based on the Example on p. 79 of the notes, we have

$$b_k = \begin{cases} \frac{3}{5}, & k \text{ a multiple of } 5 \\ \frac{1}{5}\frac{\sin(3\pi k/5)}{\sin(\pi k/5)}, & \text{otherwise.} \end{cases}$$

We also note that

$$\begin{aligned} c_k &\triangleq \frac{1}{N} \sum_{n=<N>} x_2[n] \; e^{-jk\left(\frac{2\pi}{N}\right)n} \\ &= \frac{1}{5} \sum_{n=<5>} e^{-jk\left(\frac{2\pi}{5}\right)n} \\ &= \begin{cases} 1, & k \text{ a multiple of } 5 \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

We thus obtain

$$a_k = \begin{cases} \frac{8}{5}, & k \text{ a multiple of } 5 \\ \frac{1}{5}\frac{\sin(3\pi k/5)}{\sin(\pi k/5)}, & \text{otherwise.} \end{cases}$$

Frequency response of LTI systems – Section 3.8

Continuous-time: Recall that if $x(t) = e^{st}$ is input to a LTI system with impulse response $h(t)$, the output $y(t)$ is

$$y(t) = H(s)e^{st}$$

where

$$H(s) = \int_{-\infty}^{\infty} h(t)\, e^{-st}\, dt.$$

We will be mostly interested in the special case when $s = j\omega$. The input is then the complex exponential $e^{j\omega t}$ and

$$H(j\omega) = \int_{-\infty}^{\infty} h(t)\, e^{-j\omega t}\, dt$$

is called the *frequency response* of the system.

Let $x(t)$ be a periodic signal with a Fourier series (FS) representation

$$x(t) = \sum_{k=-\infty}^{\infty} a_k \, e^{jk\omega_0 t}, \; \omega_0 = \frac{2\pi}{T} \, .$$

Then it follows that the output is given by

$$y(t) = \sum_{k=-\infty}^{\infty} a_k \, H\left(jk\omega_0\right) \, e^{jk\omega_0 t} \, .$$

Similar results hold for DT systems.

Discrete-time: Recall that if $x[n] = z^n$ is input to a LTI system with impulse response $h[n]$, the output $y[n]$ is

$$y[n] = H(z)z^n$$

where

$$H(z) = \sum_{n=-\infty}^{\infty} h[n] \; z^{-n} \; .$$

We will be mostly interested in the special case when $z = e^{j\omega}$. The input is then the complex exponential $e^{j\omega n}$ and

$$H(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h[n] \; e^{-j\omega n}$$

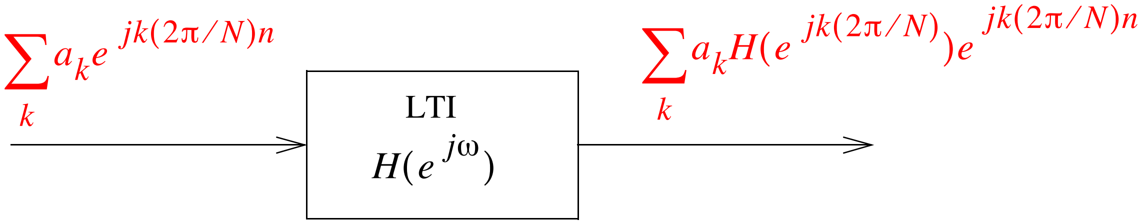
is called the *frequency response* of the system.

Let $x[n]$ be a periodic signal with a Fourier series representation

$$x[n] = \sum_{k=<N>} a_k \; e^{jk\left(\frac{2\pi}{N}\right)n}.$$

Then it follows that the output is given by

$$y[n] = \sum_{k=<N>} a_k H\left(e^{jk\left(\frac{2\pi}{N}\right)}\right) e^{jk\left(\frac{2\pi}{N}\right)n}.$$



Block diagram illustration of I/O relationship of LTI system

Example:

Recall that $x[n] = \cos \omega_0 n$ is periodic only if $\frac{2\pi}{\omega_0}$ is *rational*.

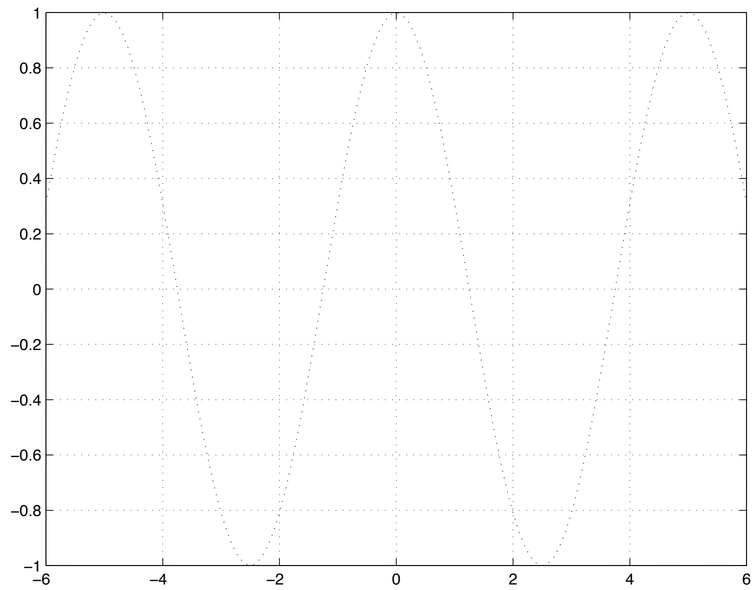
Also, $\cos\left(\frac{2\pi}{N}\right) n$ is periodic with fundamental period N.

Example:

How do $\cos\left(\frac{2\pi}{N}\right) n$ and $\cos 2\left(\frac{2\pi}{N}\right) n$ look like for $N = 5$?

n	0	1	2	3	4
$\cos \frac{2\pi n}{5}$	1	0.31	–0.81	–0.81	0.31

n	0	1	2	3	4
$\cos 2\left(\frac{2\pi n}{5}\right)$	1	–0.81	0.31	0.31	–0.81



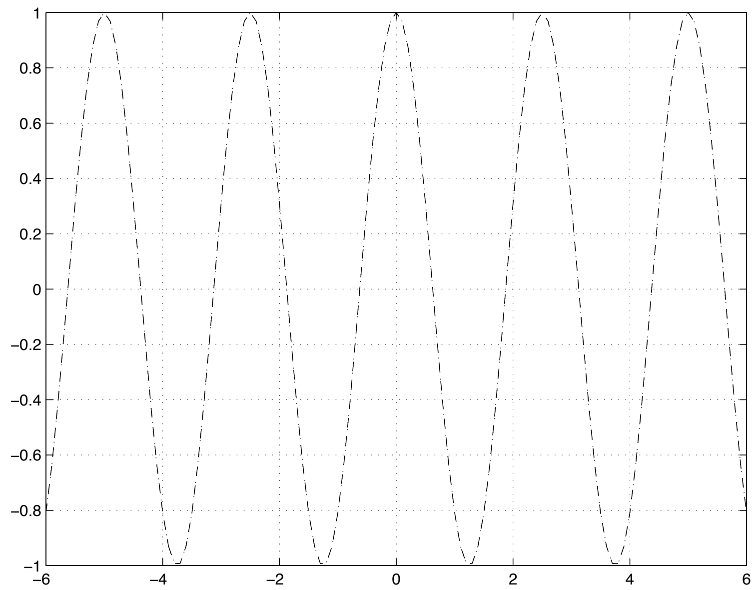
Sketch of $\cos\left(\frac{2\pi}{5}\right)n$

Note:

$$\cos\left(\frac{2\pi}{5}\right)n = \frac{1}{2}\,e^{j\left(\frac{2\pi}{5}\right)n} + \frac{1}{2}\,e^{-j\left(\frac{2\pi}{5}\right)n}.$$

A direct comparison with the DTFS synthesis equation shows that

$$a_1 = \frac{1}{2},\;\; a_{-1}\,(= a_{5-1}) = \frac{1}{2}\,.$$



Sketch of $\cos 2\left(\frac{2\pi}{5}\right)n$

Note:

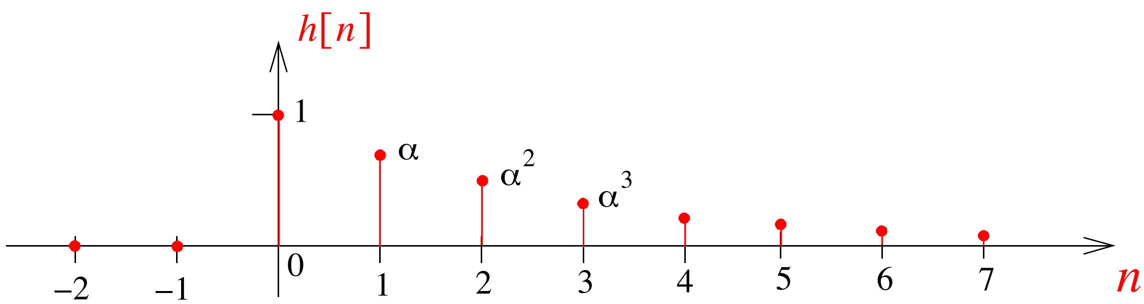
$$\cos 2\left(\frac{2\pi}{5}\right)n = \frac{1}{2}\,e^{j2\left(\frac{2\pi}{5}\right)n} + \frac{1}{2}\,e^{-j2\left(\frac{2\pi}{5}\right)n}.$$

A direct comparison with the DTFS synthesis equation shows that

$$a_2 = \frac{1}{2},\ a_{-2}\left(= a_{5-2}\right) = \frac{1}{2}\ .$$

Consider an LTI system with impulse response shown below

$$h[n] = \alpha^n \, u[n], \; |\alpha| < 1.$$



Then,

$$\begin{aligned} H(z) &\triangleq \sum_{n=-\infty}^{\infty} h[n] \, z^{-n} \\ &= \sum_{n=0}^{\infty} \alpha^n \, z^{-n} \\ &= \sum_{n=0}^{\infty} \left(\alpha \, z^{-1}\right)^n \\ &= \frac{1}{1 - \alpha \, z^{-1}} \end{aligned}$$

and

$$H(e^{j\omega}) = \frac{1}{1-\alpha\ e^{-j\omega}} \ .$$

Suppose that $x[n] = \cos\left(\frac{2\pi}{N}\right) n$ is input into such an LTI system.

Question: How can we determine the output, $y[n]$?

Since

$$\cos \frac{2\pi}{N} n = \frac{1}{2}\left(\ e^{j\frac{2\pi}{N}n} + \ e^{-j\frac{2\pi}{N}n}\right),$$

we can easily write

$$\begin{aligned} y[n] &= a_1 H\left(e^{j\frac{2\pi}{N}}\right) e^{j\frac{2\pi}{N}n} + a_{-1} H\left(e^{-j\frac{2\pi}{N}}\right) e^{-j\frac{2\pi}{N}n} \\ &= \frac{1}{2}\left[\left(\frac{1}{1-\alpha e^{-j\frac{2\pi}{N}}}\right) e^{j\frac{2\pi}{N}n} + \left(\frac{1}{1-\alpha e^{j\frac{2\pi}{N}}}\right) e^{-j\frac{2\pi}{N}n}\right]. \end{aligned}$$

Letting

$$re^{j\theta} = \frac{1}{1 - \alpha e^{-j\frac{2\pi}{N}}} \ ,$$

we have

$$y[n] = r\cos\left(\frac{2\pi}{N}n + \theta\right) \ .$$

Question: How can we determine r and θ?

$$re^{j\theta} \ = \ \frac{1}{1 - \alpha\cos\frac{2\pi}{N} + j\alpha\sin\frac{2\pi}{N}}$$

so that

$$r \ = \ \frac{1}{\sqrt{(1 - 2\alpha\cos\frac{2\pi}{N}) + \alpha^2}}$$

$$\tan\theta \ = \ \frac{-\alpha\sin\frac{2\pi}{N}}{1 - \alpha\cos\frac{2\pi}{N}} \ .$$