Discrete-time Fourier transform (DTFT) representation of DT aperiodic signals – Section 5.1[3]

The (DT) Fourier transform (or spectrum) of $x[n]$ is

$$\boxed{X\left(e^{j\omega}\right) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}}$$

$x[n]$ can be reconstructed from its spectrum using the inverse Fourier transform

$$\boxed{x[n] = \frac{1}{2\pi}\int_{2\pi} X\left(e^{j\omega}\right) e^{j\omega n}\, d\omega}$$

The above two equations are referred to as the *analysis equation* and the *synthesis equation* respectively.

[3] These slides are based on lecture notes of Professors L. Lampe, C. Leung, and R. Schober

Notation:

$$\begin{aligned} X\left(e^{j\omega}\right) &= \mathcal{F}\{x[n]\} \\ x[n] &= \mathcal{F}^{-1}\{X\left(e^{j\omega}\right)\} \end{aligned}$$

$x[n]$ and $X\left(e^{j\omega}\right)$ form a Fourier transform pair, denoted by

$$x[n] \leftrightarrow X\left(e^{j\omega}\right)$$

Main differences from CTFT

1. $X\left(e^{j\omega}\right)$ is periodic with period 2π since $e^{-j\omega n}$ is periodic in ω with period 2π.

2. As a result of the above-mentioned periodicity, the integration for the synthesis equation is over a finite interval (of length 2π).

Example: Determine the DTFT of $h[n] = a^n \; u[n], |a| < 1.$

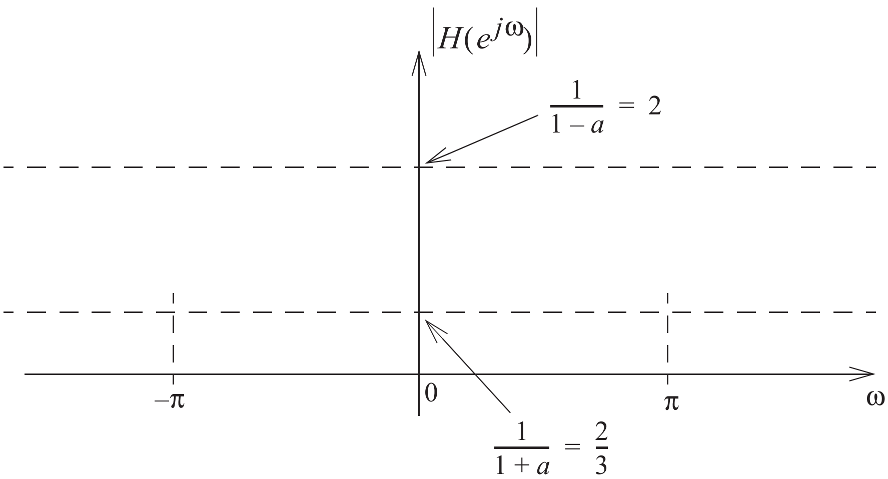
Then,

$$\begin{aligned} H\left(e^{j\omega}\right) &= \sum_{n=0}^{\infty} a^n \; e^{-j\omega n} \\ &= \sum_{n=0}^{\infty} \left(a \; e^{-j\omega}\right)^n \\ &= \frac{1}{1 - a \; e^{-j\omega}} \end{aligned}$$

Hence,

$$\begin{aligned} |H\left(e^{j\omega}\right)| &= \frac{1}{\sqrt{(1 - a\cos\omega)^2 + (a\sin\omega)^2}} \\ &= \frac{1}{\sqrt{1 - 2a\cos\omega + a^2}} \end{aligned}$$

A plot of $|H\left(e^{j\omega}\right)|$ for $a = 0.5$ is shown in the figure below.



Since

$$H\left(e^{j\omega}\right) = \frac{1}{1-a\ e^{-j\omega}}$$
$$= \frac{1}{1-a\ e^{-j\omega}} \times \frac{1-a\ e^{j\omega}}{1-a\ e^{j\omega}}$$
$$= \frac{1-a\ e^{j\omega}}{1-2a\cos\omega+a^2}$$

and

$$1-a\ e^{j\omega} = 1-a\cos\omega - ja\sin\omega$$

we have

$$\angle H\left(e^{j\omega}\right) = \tan^{-1}\frac{-a\sin\omega}{1-a\cos\omega}$$

Example: Determine the DTFT of the rectangular pulse

$$x[n] = \begin{cases} 1, & |n| \leq N \\ 0, & \text{otherwise.} \end{cases}$$

Then,

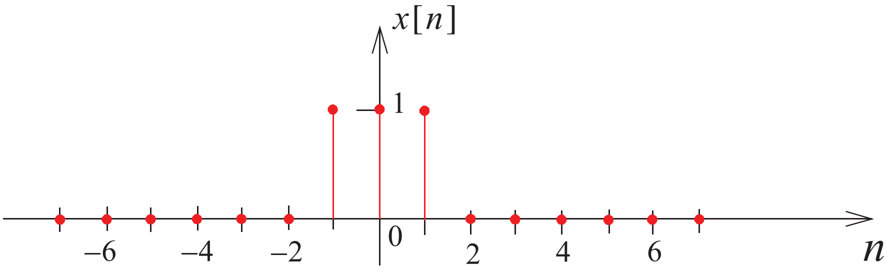
$$\begin{aligned}
X\left(e^{j\omega}\right) &= \sum_{n=-N}^{N} e^{-j\omega n} \\
&= \sum_{m=0}^{2N} e^{-j\omega(m-N)} \\
&\uparrow \\
(\text{using } m &= n+N) \\
&= \frac{e^{j\omega N}\left(1-e^{-j\omega(2N+1)}\right)}{1-e^{-j\omega}} \\
&= \frac{e^{j\omega N} e^{-j\omega\left(N+\frac{1}{2}\right)}\left\{e^{j\omega\left(N+\frac{1}{2}\right)}-e^{-j\omega\left(N+\frac{1}{2}\right)}\right\}}{e^{-j\frac{\omega}{2}}\left\{e^{j\frac{\omega}{2}}-e^{-j\frac{\omega}{2}}\right\}} \\
&= \frac{\left\{e^{j\omega\left(N+\frac{1}{2}\right)}-e^{-j\omega\left(N+\frac{1}{2}\right)}\right\}}{\left\{e^{j\frac{\omega}{2}}-e^{-j\frac{\omega}{2}}\right\}}
\end{aligned}$$

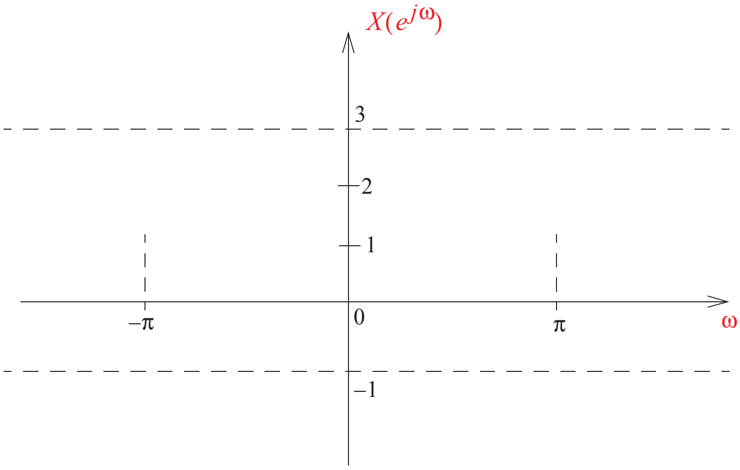
$$
\begin{aligned}
&= \frac{2j\sin\left[\omega\left(N+\frac{1}{2}\right)\right]}{2j\sin\left[\omega/2\right]}\\
&= \frac{\sin\left[\omega\left(N+\frac{1}{2}\right)\right]}{\sin\left[\omega/2\right]}
\end{aligned}
$$

For $N = 1$,

$$
\begin{aligned}
X\left(e^{j\omega}\right) &= \frac{\sin\left[\omega+\frac{\omega}{2}\right]}{\sin\left[\frac{\omega}{2}\right]}\\
&= \frac{\sin\omega\cos\frac{\omega}{2}+\cos\omega\sin\frac{\omega}{2}}{\sin\frac{\omega}{2}}\\
&= \frac{2\sin\frac{\omega}{2}\cos\frac{\omega}{2}\cos\frac{\omega}{2}+\cos\omega\sin\frac{\omega}{2}}{\sin\frac{\omega}{2}}\\
&= 2\cos^2\frac{\omega}{2}+\cos\omega\\
&= 1+2\cos\omega
\end{aligned}
$$

Sketch $X\left(e^{j\omega}\right)$ for $N = 1$ in the figure below.





DTFT convergence issues

Recall the DTFT analysis and synthesis equations

$$X\left(e^{j\omega}\right) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} \tag{4}$$

$$x[n] = \frac{1}{2\pi}\int_{2\pi} X\left(e^{j\omega}\right) e^{j\omega n}\, d\omega \tag{5}$$

The analysis equation (4) contains an infinite sum. Sufficient conditions to guarantee convergence are:

(1) $x[n]$ is *absolutely summable*, i.e.

$$\sum_{n=-\infty}^{\infty} |x[n]| < \infty$$

or (2) $x[n]$ has *finite energy*, i.e.

$$\sum_{n=-\infty}^{\infty} |x[n]|^2 < \infty$$

There are no convergence issues associated with the synthesis equation (5) since the integral is over a finite interval. For example, unlike the CTFT case, the Gibbs phenomenon is absent when (5) is used.

DTFT for periodic signals – Section 5.2

Similar to the CT case, we can express the FT of DT periodic signals in terms of impulses. It can be shown (see (5.19) and (5.20) in textbook) that if the DT periodic signal $\tilde{x}[n]$ of fundamental period N has as DTFS coefficients a_k, i.e.

$$\tilde{x}[n] = \sum_{k=<N>} a_k \, e^{jk\left(\frac{2\pi}{N}\right)n}$$

then

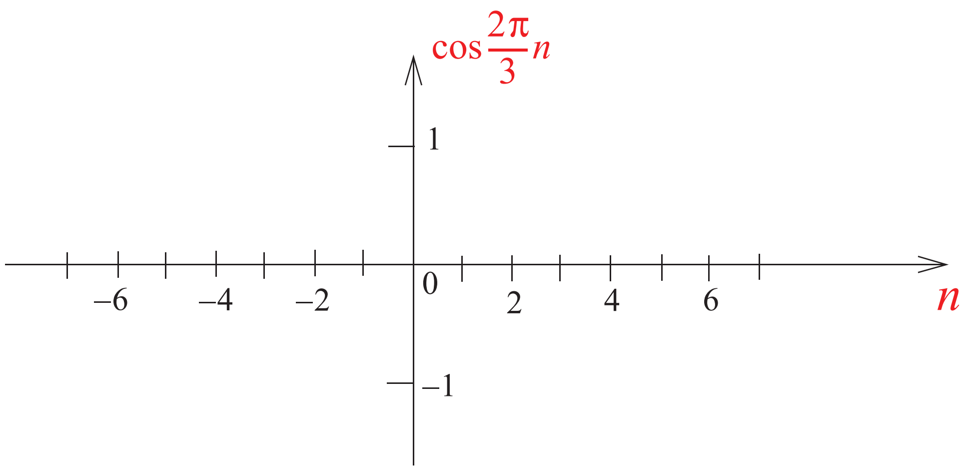
$$\tilde{X}\left(e^{j\omega}\right) = \sum_{k=-\infty}^{\infty} 2\pi a_k \, \delta\left(\omega - \frac{2\pi k}{N}\right). \quad (6)$$

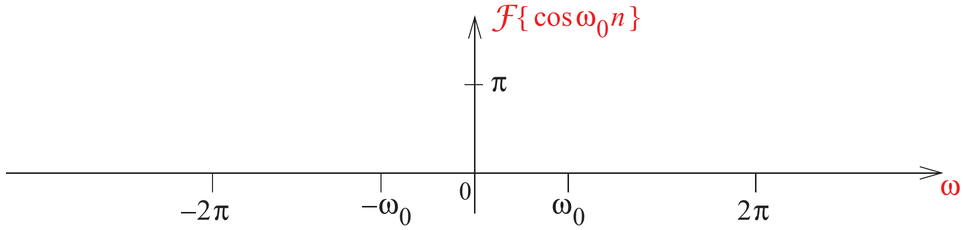
Furthermore, if we define $x[n]$ as one basic period of $\tilde{x}[n]$, it can be shown that

$$a_k = \frac{1}{N} X\left(e^{jk\frac{2\pi}{N}}\right).$$

In words, (6) states that the DTFT of $\tilde{x}[n]$ is a sequence of impulses located at multiples of the fundamental frequency $\frac{2\pi}{N}$; the strength of the impulse located at $\omega = k\frac{2\pi}{N}$ is $2\pi a_k$.

Example: $x[n] = \cos(\omega_0 n)$, where $\omega_0 = \frac{2\pi}{3}$.





$$\cos \omega_0 n = \frac{1}{2}\, e^{j\omega_0 n} + \frac{1}{2}\, e^{-j\omega_0 n}$$

Fact:

$$e^{j\omega_0 n} \leftrightarrow 2\pi \sum_{l=-\infty}^{\infty} \delta\left(\omega - \omega_0 - 2\pi l\right)$$

Proof:

The inverse DTFT of the RHS is

$$\begin{aligned}
&\frac{1}{2\pi} \int_{2\pi} \left[2\pi \sum_{l=-\infty}^{\infty} \delta\left(\omega - \omega_0 - 2\pi l\right) e^{j\omega n} \right] d\omega \\
&= \sum_{l=-\infty}^{\infty} \left[\int_{2\pi} \delta\left(\omega - \omega_0 - 2\pi l\right) e^{j\omega n} \, d\omega \right] \\
&= e^{j\omega_0 n} .
\end{aligned}$$

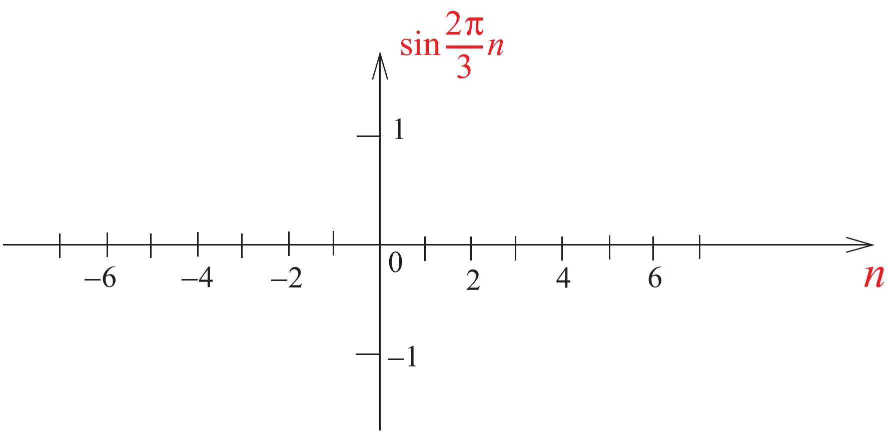
The last line follows since only one impulse is included in any interval of length 2π. Hence,

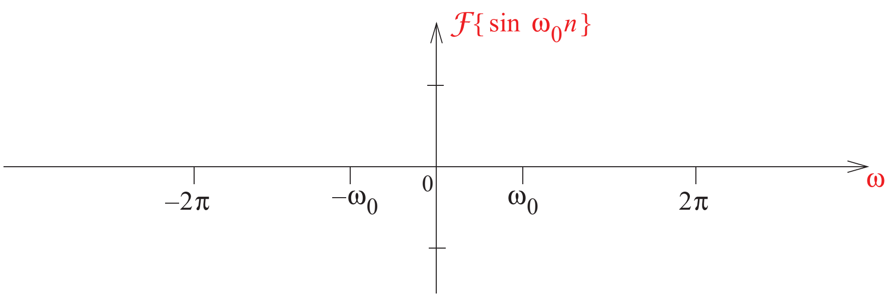
$$\begin{aligned}
\cos \frac{2\pi}{3} n \quad \leftrightarrow \quad \pi & \left[\sum_{l=-\infty}^{\infty} \delta\left(\omega - \frac{2\pi}{3} - 2\pi l\right) \right. \\
& + \left. \sum_{l=-\infty}^{\infty} \delta\left(\omega + \frac{2\pi}{3} - 2\pi l\right) \right]
\end{aligned}$$

Exercise: Show that

$$\sin\frac{2\pi}{3}n \quad \leftrightarrow \quad \frac{\pi}{j}\left[\sum_{l=-\infty}^{\infty}\delta\left(\omega-\frac{2\pi}{3}-2\pi l\right) - \sum_{l=-\infty}^{\infty}\delta\left(\omega+\frac{2\pi}{3}-2\pi l\right)\right]$$

Plot $\sin\left(\frac{2\pi}{3}n\right)$ and its DTFT in the figures below.





Example: Determine the DTFT of the impulse train

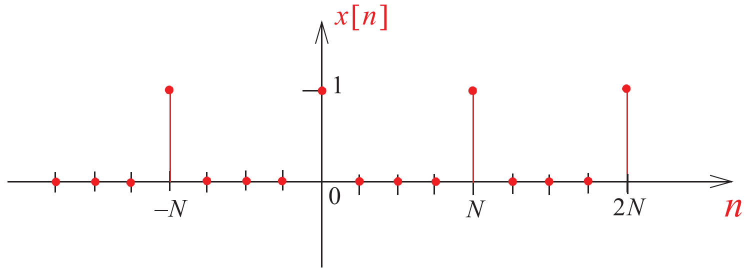
$$x[n] = \sum_{k=-\infty}^{\infty} \delta[n - kN].$$

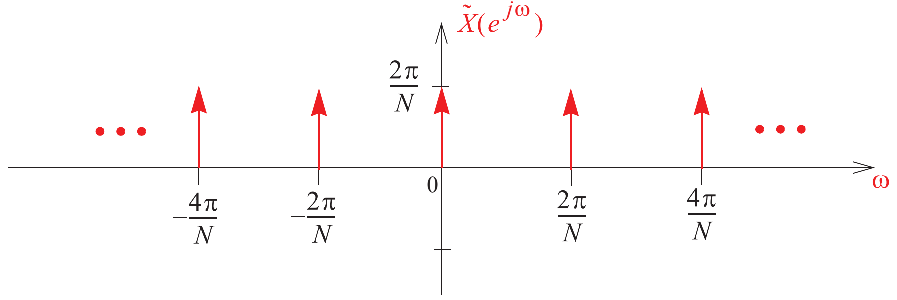
Then,

$$a_k = \frac{1}{N} \sum_{n=<N>} x[n]\, e^{-jk\left(\frac{2\pi}{N}\right)n} = \frac{1}{N}$$

and from (6), we have

$$\tilde{X}\left(e^{j\omega}\right) = \frac{2\pi}{N} \sum_{k=-\infty}^{\infty} \delta\left(\omega - \frac{2\pi k}{N}\right).$$





Some properties of the DTFT – Sections 5.3-5.7

1. *Periodicity – Section 5.3.1*

   The DTFT is *always periodic* in ω with period 2π, i.e.

   $$X\left(e^{j(\omega+2\pi)}\right) = X\left(e^{j\omega}\right)$$

   In contrast, the CTFT is *not generally* periodic.

2. *Linearity – Section 5.3.2 – similar to CTFT*

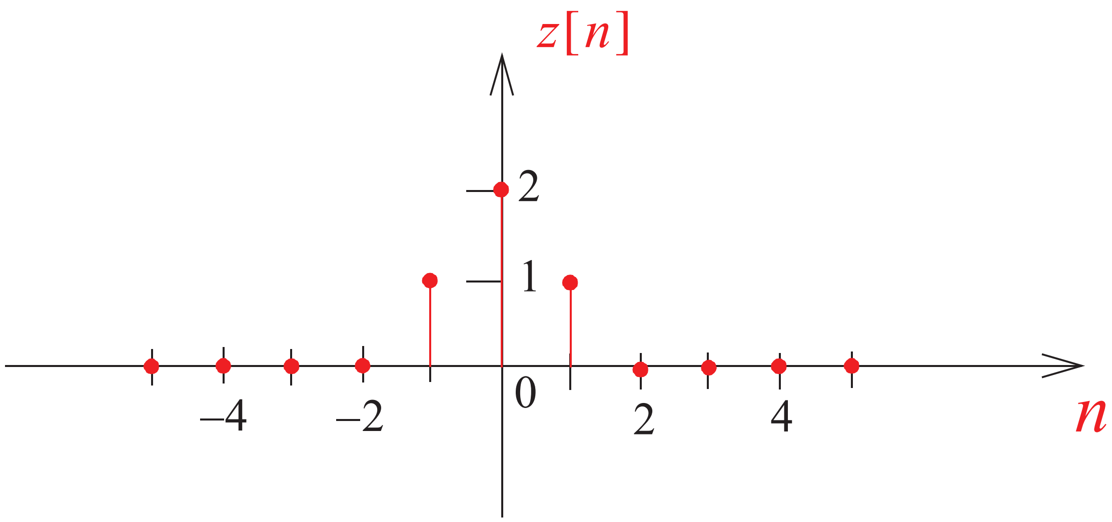
   Let $x[n] \leftrightarrow X\left(e^{j\omega}\right)$, $y[n] \leftrightarrow Y\left(e^{j\omega}\right)$.

   Then,

   $$\begin{aligned} z[n] &= ax[n] + by[n] \\ \leftrightarrow Z\left(e^{j\omega}\right) &= aX\left(e^{j\omega}\right) + bY\left(e^{j\omega}\right) \end{aligned}$$

Proof: follows directly from the defn. of the DTFT.

Example: What is the DTFT of the signal, $z[n]$, shown below?



3. *Time-shift – Section 5.3.3 – similar to CTFT*

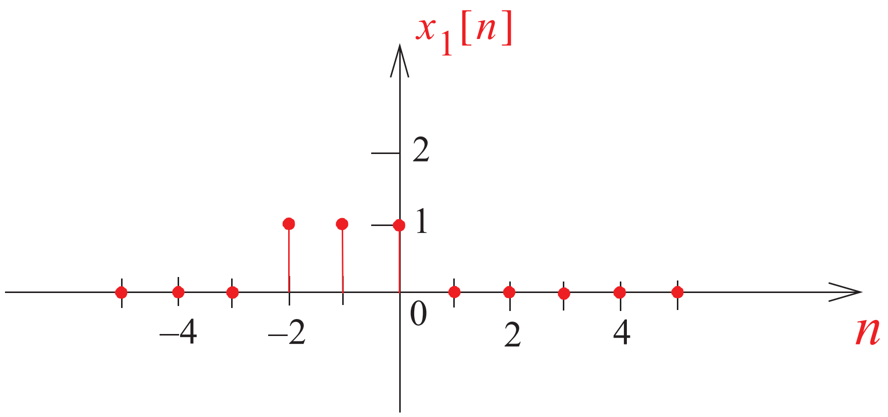
If $x[n] \leftrightarrow X\left(e^{j\omega}\right)$, then

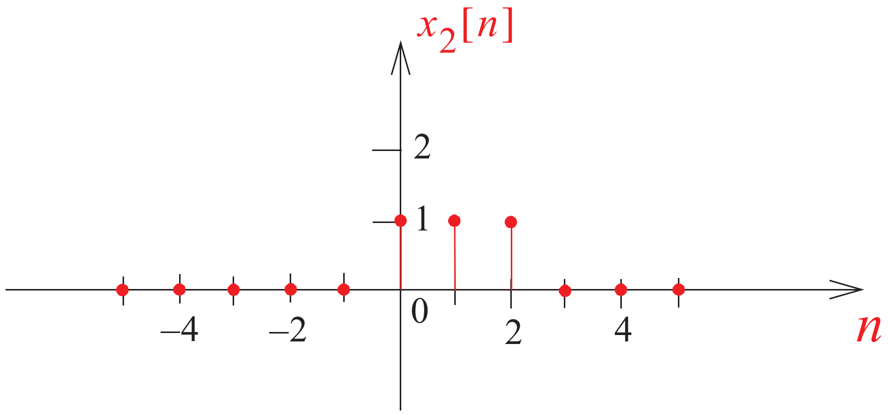
$$x[n-N] \leftrightarrow e^{-j\omega N} X\left(e^{j\omega}\right) .$$

Proof:

Note that when a signal is delayed by N, the *spectrum amplitude* is *unchanged* whereas the spectrum *phase* is changed by $-\omega N$.

Example: What are the DTFT's of the signals shown below?





4. *Conjugation – Section 5.3.4 – similar to CTFT*

If $x[n] \leftrightarrow X\left(e^{j\omega}\right)$, then

$$x^*[n] \leftrightarrow X^*\left(e^{-j\omega}\right) .$$

Proof:

As a result, if $x[n]$ is real, we have

$$X\left(e^{-j\omega}\right) = X^*\left(e^{j\omega}\right)$$

i.e., the spectrum *magnitude* is an *even* function of ω and the spectrum *phase* is an *odd* function of ω.

5. *Differencing and Accumulation – Section 5.3.5*

Differencing: Using the *linearity* and *time-shift properties*, if $x[n] \leftrightarrow X\left(e^{j\omega}\right)$, then the DTFT of the first difference signal is given by

$$x[n] - x[n-1] \leftrightarrow \left(1 - e^{-j\omega}\right) X\left(e^{j\omega}\right) .$$

Accumulation:

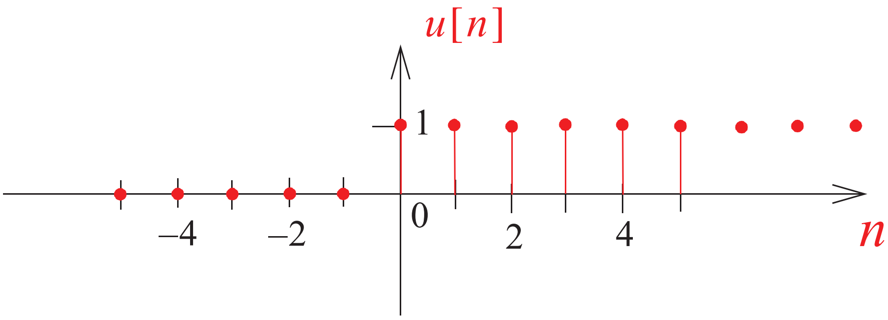
$$\text{Let } y[n] = \sum_{m=-\infty}^{n} x[m] .$$

Then,

$$\begin{aligned} Y\left(e^{j\omega}\right) &= \frac{X\left(e^{j\omega}\right)}{1 - e^{-j\omega}} \\ &\quad + \pi X\left(e^{j0}\right) \sum_{k=-\infty}^{\infty} \delta(\omega - 2\pi k) . \end{aligned}$$

Example: Determine the DTFT of the unit step sequence

$$u[n] = \begin{cases} 0, & n < 0 \\ 1, & n \geq 0. \end{cases}$$



6. *Parseval's relation – Section 5.3.9 – similar to CTFT*

If $x[n] \leftrightarrow X\left(e^{j\omega}\right)$, then

$$\sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi}\int_{2\pi} \left|X\left(e^{j\omega}\right)\right|^2 \, d\omega. \quad (7)$$

The LHS of (7) represents the *total energy* in the signal $x[n]$. The integrand, $\left|X\left(e^{j\omega}\right)\right|^2$, in the RHS is the *energy density spectrum* of $x[n]$.

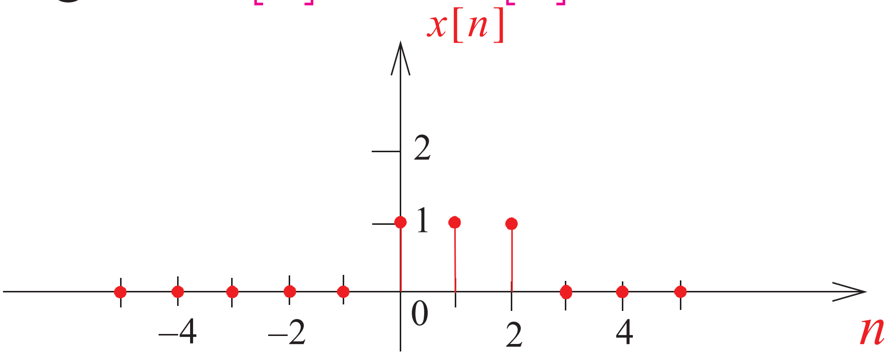
7. *Convolution – Section 5.4 – similar to CTFT*

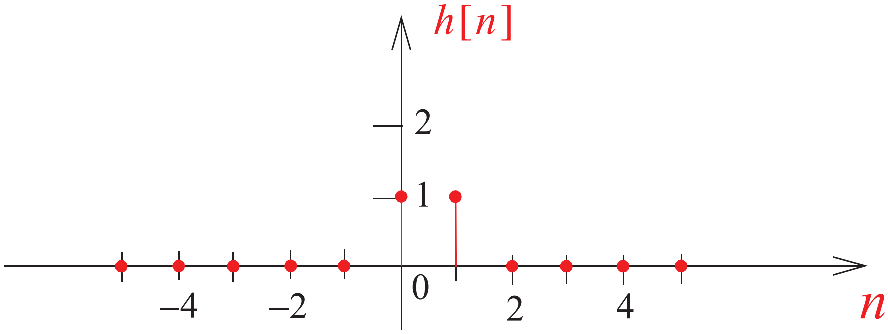
$$\text{If} \quad y[n] = x[n] * h[n]$$

$$\text{then} \quad Y\left(e^{j\omega}\right) = X\left(e^{j\omega}\right) H\left(e^{j\omega}\right) \; .$$

Proof:

Example: Determine the convolution of the DT signals $x[n]$ and $h[n]$ shown below.





8. *Multiplication – Section 5.5*

$$\text{If} \quad y[n] = a[n]b[n]$$

then

$$Y\left(e^{j\omega}\right) = \frac{1}{2\pi}\int_{2\pi} A\left(e^{j\theta}\right) B\left(e^{j(\omega-\theta)}\right) d\theta \ .$$

The RHS of the previous equation is referred to as the periodic convolution of $A\left(e^{j\omega}\right)$ and $B\left(e^{j\omega}\right)$.

Proof: