HW #1 Solutions

1 Spring-mass damper w/ dead zone

1. $Q = \{q_0, q_+, q_-\}$

$$f = \left\{ \begin{bmatrix} x_2 \\ -bx_2 \end{bmatrix}, \begin{bmatrix} x_2 \\ -bx_2 - kx_1 + k\epsilon \end{bmatrix}, \begin{bmatrix} x_2 \\ -bx_2 - kx_1 - k\epsilon \end{bmatrix} \right\}$$

$$\mathrm{Dom} = \{q_0\} \times \{x \mid -\epsilon \le x_1 \le \epsilon\} \cup \{q_+\} \times \{x \mid x_1 \ge \epsilon\} \cup \{q_-\} \times \{x \mid x_1 \le -\epsilon\}$$

$\mathrm{Init} = \mathrm{Dom}$, $\Sigma = \{\emptyset\}$.

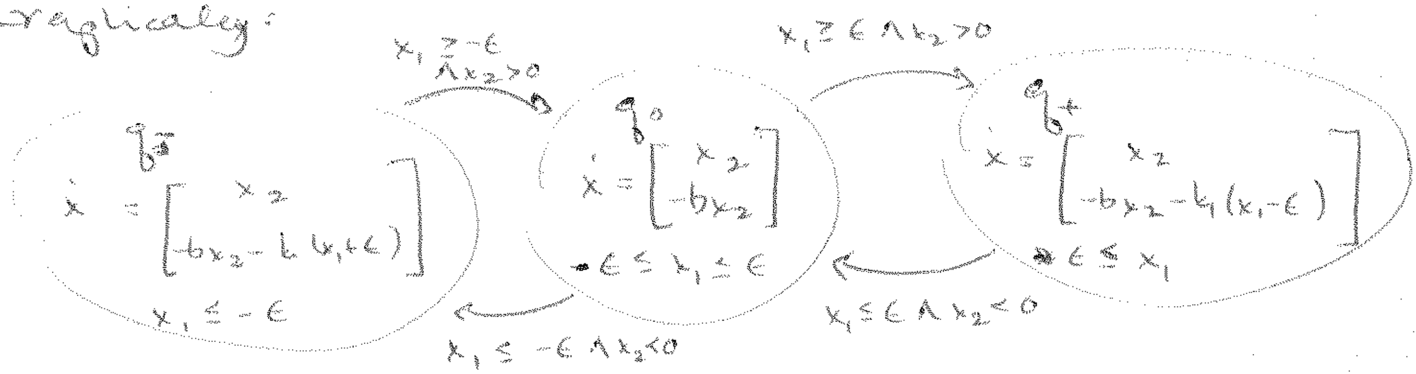
$\mathrm{Guard}(q_0, q_+) = \{x \mid x_1 \ge \epsilon \wedge x_2 > 0\}$ | $R(q_0, \mathrm{Guard}(q_0,q_+)) = (q_+, x)$

$\mathrm{Guard}(q_0, q_-) = \{x \mid x_1 \le -\epsilon \wedge x_2 < 0\}$ | $R(q_0, \mathrm{Guard}(q_0,q_-)) = (q_-, x)$

$\mathrm{Guard}(q_+, q_0) = \{x \mid x_1 \le \epsilon \wedge x_2 < 0\}$ | $R(q_+, \mathrm{Guard}(q_+,q_0)) = (q_0, x)$

$\mathrm{Guard}(q_-, q_0) = \{x \mid x_1 \ge -\epsilon \wedge x_2 > 0\}$ | $R(q_-, \mathrm{Guard}(q_-,q_0)) = (q_0, x)$

Graphically:

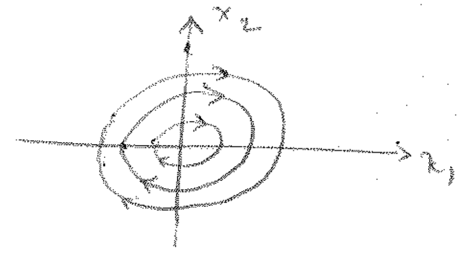


2. a) $m=1, b=0, k=2$

undamped

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -2 & 0 \end{bmatrix} x \Longrightarrow \lambda_{1,2} = \pm j\sqrt{2}$$

Center

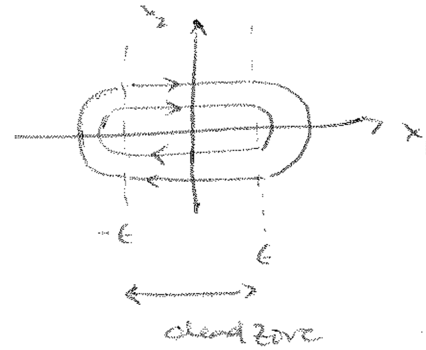


w/out dead zone

In the dead zone,

$$\dot{x} = \begin{bmatrix} x_2 \\ 0 \end{bmatrix}$$

Outside the dead zone, dynamics are similar to those of a center, but x_1 is offset by $\pm\epsilon$.



b) $m=1$, $b=1$, $k=2$

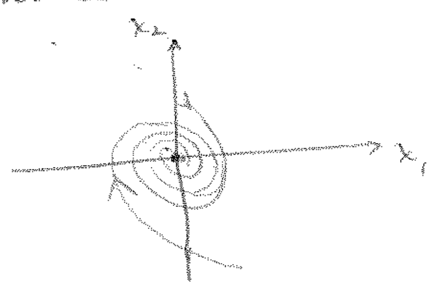
underdamped.

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -2 & -1 \end{bmatrix} \implies \lambda(\lambda+1)+2=0$$

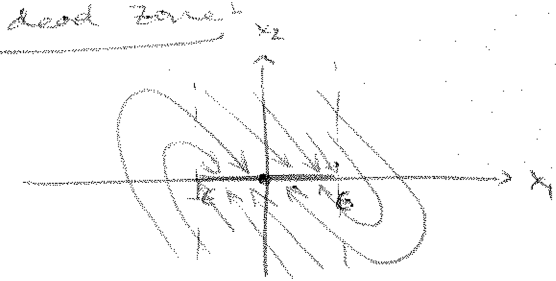
$$\lambda_{1,2} = \frac{-1 \pm \sqrt{1-4\cdot 2}}{2} = -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$$

w/out dead zone:

stable focus



w/ dead zone:



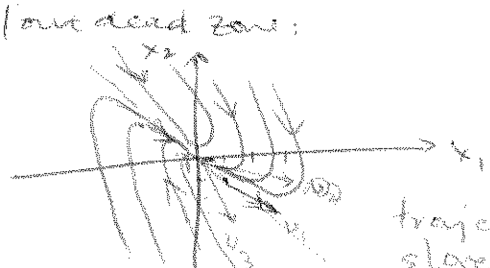
in dead zone, $\dot{x} = \begin{bmatrix} x_2 \\ -bx_2 \end{bmatrix}$ linear w/ slope $-b=-1$ between $|x_1| \le \epsilon$.

c) $m=1$, $b=3$, $k=2$. ~~overdamped~~ overdamped.

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \implies \lambda(\lambda+3)+2=0$$

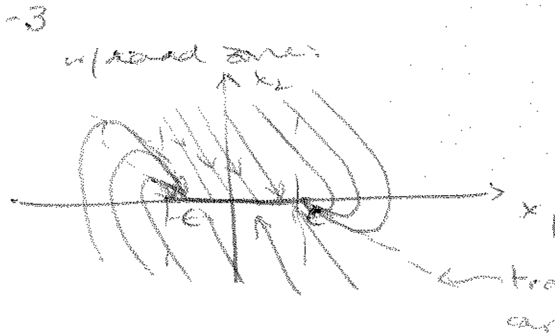
$$\lambda_{1,2} = -1, -3$$

w/out dead zone:



trajectories start w/ slope -3, end w/ slope tangential to -1.

w/ dead zone:



trajectories arrive tangent to line w/ slope -1.

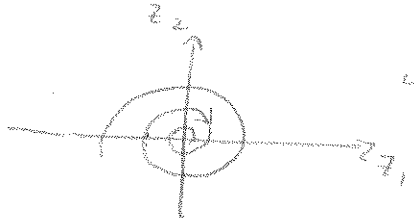
stable node.

$A v_1 = \lambda_1 v_1$, $v_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, $A v_2 = \lambda_2 v_2$, $v_2 = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$

$v_{12} = -v_{11}$, $v_{22} = -3 \cdot v_{21}$

3. underdamped: $m=1$, $b=1$, $k=2$

$A = M^{-1} J M$, where $J = \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix} = \begin{bmatrix} -1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{bmatrix}$
(Jordan real form)

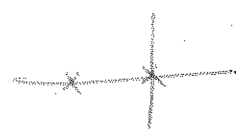


where trajectories follow
$\dot{r} = \alpha r$, $\dot{\theta} = \beta$
w/ $r = \sqrt{z_1^2 + z_2^2}$, $\theta = \tan^{-1}(z_2/z_1)$

4. $0 = s^2 + \frac{b}{m} s + \frac{k}{m}$ (characteristic equation)

$$s = \frac{-\frac{b}{m} \pm \sqrt{\left(\frac{b}{m}\right)^2 - 4\frac{k}{m}}}{2}$$

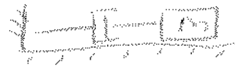
Since $m, k, b > 0$ and are real,



a) For one eigenvalue at 0,

$-\frac{b}{m} + \sqrt{\left(\frac{b}{m}\right)^2 - 4\left(\frac{k}{m}\right)} = 0 \Rightarrow k = 0$. (no spring)

This system will have one pole at 0, & the other at $-\frac{b}{2m}$. Physically, this corresponds to a system w/ a damper but no spring:



b) Both eigval. at 0,

$k=0$ (as above), and $-\frac{b}{m} - \frac{b}{m} = 0 \Rightarrow b = 0$.

Physically, this system is a free standing mass:

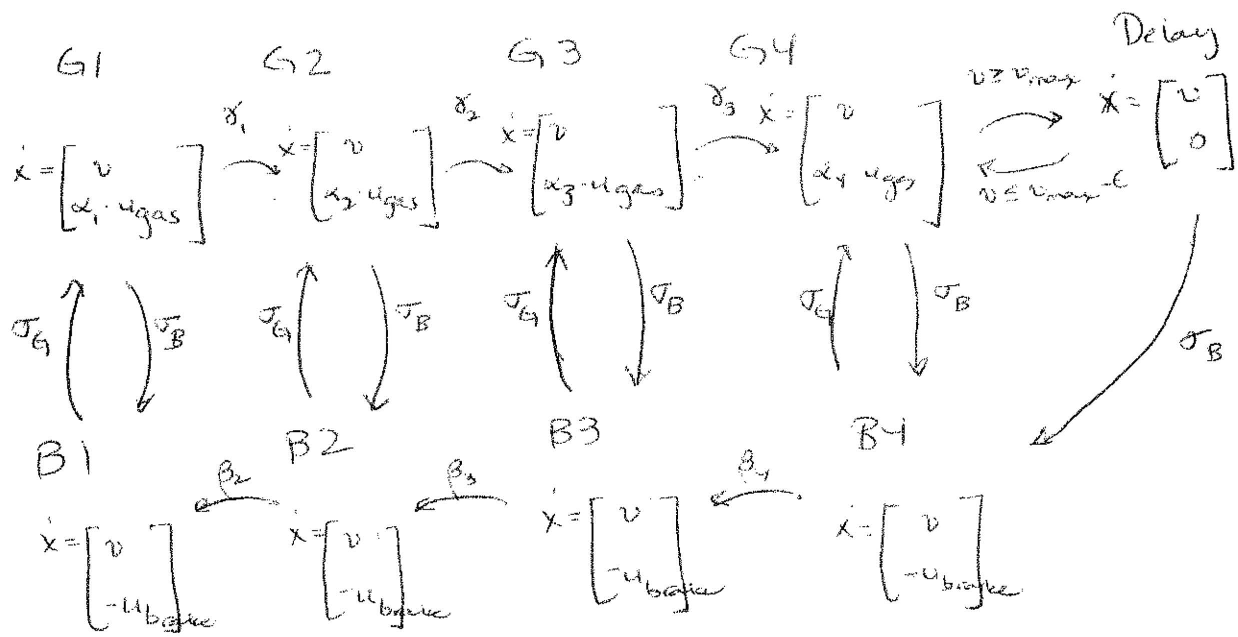
Therefore we know that all spring-mass-damper systems will be asymptotically stable for $m, b, k > 0$.

Problem 2, HW #1

There are many different solutions.
Most people included 4 "brake" modes & included some sort of constraints on the continuous control law so that the driver must use no brake pedal while driving & no gas pedal while braking.

($\epsilon > 0$, $v_{max} > 0$)

Assume graphic is for v, $\alpha_i(v)$ & starts at $v = 0$.



$Q = \{G1, G2, G3, G4, B1, B2, B3, B4\}$

$\Sigma = \{\sigma_B, \sigma_G\}$ (switching foot from gas pedal to brake pedal, & from brake pedal to gas pedal, respectively)

$U = \{u_{gas}, u_{brake}\} \in \{[0, \bar{u}_{gas}] \times [\bar{u}_{brake}, 0]\}$ (continuous input)

$X = \{x, v\} \in \mathbb{R}^2 \geq 0$.

$R(G_i, x, \sigma_B, u) = (B_i, x)$ $\quad i \in \{1,2,3,4\}$

$R(B_i, x, \sigma_G, u) = (G_i, x)$ $\quad i \in \{1,2,3,4\}$

$R(G_i, \text{Guard}(G_i, G_{i+1}), \{\phi\}, u) = (G_{i+1}, x)$ $\quad i \in \{1,2,3\}$

$R(B_i, \text{Guard}(B_i, B_{i-1}), \{\phi\}, u) = (B_{i-1}, x)$ $\quad i \in \{2,3,4\}$

$\gamma_i = \text{Guard}(G_i, G_{i+1}) = \{x \mid \alpha_i(v) \leq 0.6 \wedge \alpha_{i+1}(v) \geq 0.6\}$ $\quad i \in \{1,2,3\}$

$\beta_i = \text{Guard}(B_i, B_{i-1}) = \{x \mid \alpha_i(v) \leq 0.3 \wedge \alpha_{i-1}(v) \geq 0.3\}$ $\quad i \in \{2,3,4\}$

$$\begin{aligned}
\text{Dom} = & \bigcup_{i \in \{1,2,3,4\}} G_i \times \{x \mid \alpha_i(v) \geq 0.6 \wedge v \geq 0\} \times U \times \{\phi, \sigma_B\} \\
& \bigcup_{i \in \{2,3,4\}} B_i \times \{x \mid \alpha_i(v) \geq 0.3 \wedge v \geq 0\} \times U \times \{\phi, \sigma_G\} \\
& \cup \text{Delay} \times \{x \mid \alpha_4(v) \geq 0.6 \wedge v = v_{max}\} \times U \times \{\phi, \sigma_B\} \\
& \cup B_1 \times \{x \mid \alpha_i(v) \geq 0.6 \wedge v \geq 0\} \times \{ \times U \times \{\phi, \sigma_G\}
\end{aligned}$$

Init = Dom.

Use standard tests on Reach + Trans to show system is non-blocking + deterministic.

Problem #3, HW #1

$$\dot{x}_1 = \operatorname{sgn}(x_1) + 2\operatorname{sgn}(x_2)$$
$$\dot{x}_2 = -2\operatorname{sgn}(x_1) - \operatorname{sgn}(x_2)$$

q_2: $\dot{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ | q_1: $\dot{x} = \begin{bmatrix} 3 \\ -3 \end{bmatrix}$

q_3: $\dot{x} = \begin{bmatrix} -3 \\ 3 \end{bmatrix}$ | q_4: $\dot{x} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$

$$Q = \{q_1, q_2, q_3, q_4\}$$
$$X = \mathbb{R}^2$$

$$\begin{aligned}
\text{Dom} = \; & q_1 \times \{x \mid x_1 > 0 \wedge x_2 > 0\} \cup \\
& q_2 \times \{x \mid x_1 < 0 \wedge x_2 > 0\} \cup \\
& q_3 \times \{x \mid x_1 < 0 \wedge x_2 < 0\} \cup \\
& q_4 \times \{x \mid x_1 > 0 \wedge x_2 < 0\}
\end{aligned}$$

$$R(q_1, \{x \mid x_1 > 0 \wedge x_2 = 0^+\}) = \left(q_4, \begin{bmatrix} x_1 \\ 0^- \end{bmatrix}\right)$$
$$R(q_2, \{x \mid x_1 = 0^- \wedge x_2 > 0\}) = \left(q_1, \begin{bmatrix} 0^+ \\ x_2 \end{bmatrix}\right)$$
$$R(q_3, \{x \mid x_1 < 0 \wedge x_2 = 0^-\}) = \left(q_2, \begin{bmatrix} x_1 \\ 0^+ \end{bmatrix}\right)$$
$$R(q_4, \{x \mid x_1 = 0^+ \wedge x_2 < 0\}) = \left(q_3, \begin{bmatrix} 0^- \\ x_2 \end{bmatrix}\right)$$

where $0^+ := \min_{x>0} x$

$0^- := \max_{x<0} x$

Domain is given, $\therefore$ $\text{Trans} = \text{Dom}^c = q_1 \times \{x \mid x_1 \le 0 \text{ or } x_2 \le 0\} \cup$
$q_2 \times \{x \mid x_1 \ge 0 \text{ or } x_2 \le 0\} \cup$
$q_3 \times \{x \mid x_1 \ge 0 \text{ or } x_2 \ge 0\} \cup$
$q_4 \times \{x \mid x_1 \le 0 \text{ or } x_2 \ge 0\}$

~~From Ian Starnes:~~

τ_0 is a finite execution time which depends on the initial state.

Case 1. $x_1(0)\, x_2(0) > 0$

$$\tau_0 = \frac{|x_2(0)|}{3}$$

Case 2: $x_1(0)\, x_2(0) < 0$

$$\tau_0 = |x_1(0)|$$

All subsequent intervals have fixed duration

$$\tau' = |x_1(0)| + |x_2(0)|$$

or $\tau'' = \frac{1}{3}\left(|x_1(0)| + |x_2(0)|\right)$

The sequence of subsequent intervals alternate between τ' & τ''. Let $\tau_i = \begin{cases} \tau' & \text{for } i \text{ odd} \\ \tau'' & \text{for } i \text{ even} \end{cases}$,

Then

$$\|\tau_n\| = |\tau_0| + |\tau_1| + \ldots + |\tau_n|$$

$$= |\tau_0| + \sum_{i \text{ odd}}^{n} |\tau'| + \sum_{i \text{ even}}^{n} |\tau''|$$

$$= |\tau_0| + |\tau'| \cdot \tfrac{n}{2} + |\tau''|\, n/2$$

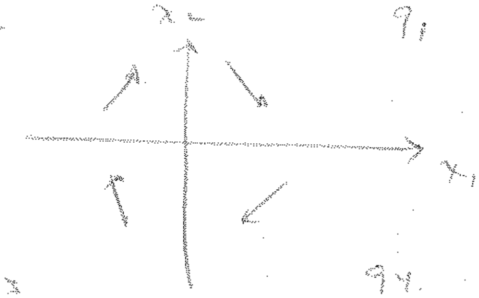
$$\lim_{n\to\infty} \|\tau_n\| = \lim_{n\to\infty} \left(|\tau_0| + \tfrac{n}{2}|\tau'| + \tfrac{n}{2}|\tau''|\right) = \infty$$

Therefore hybrid system is NOT zeno for IC $x(0) \neq 0$

For the second case (corrected version), replace $f(\cdot,\cdot)$ with:

$$f_1(x) = \begin{bmatrix} 1 \\ -3 \end{bmatrix}, \quad f_2(x) = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

$$f_3(x) = \begin{bmatrix} -1 \\ 3 \end{bmatrix}, \quad f_4(x) = \begin{bmatrix} -3 \\ -1 \end{bmatrix}$$



2 A system is Zeno when an infinite number of switches occur in finite time. ~~meaning~~ that A system in which dwell time in *all* modes is finite will be non-Zeno, since an infinite # of switches will only occur in an infinite amount of time. So to show non-Zeno-ness we merely need to demonstrate that the system will have finite dwell time in each mode.

Assume we start in q_1 from $x = \begin{bmatrix} a \\ b \end{bmatrix}$. When we switch to mode q_4, the state will be $x = \begin{bmatrix} c \\ 0 \end{bmatrix}$, where we can write c in terms of $a+b$. Then to switch to mode q_3, $x = \begin{bmatrix} 0 \\ 3c \end{bmatrix}$. To subsequently switch to q_2, $x = \begin{bmatrix} -c \\ 0 \end{bmatrix}$. Then to subsequently switch to q_1, $x = \begin{bmatrix} +3c \\ 0 \end{bmatrix}$, then again we arrive at q_4 w/ $x = \begin{bmatrix} c \\ 0 \end{bmatrix}$.

Since $\dot{x}$ = constant, we can compute the time t taken to get to each new mode: $\tau = c/3$. Therefore in one cycle, $\tau_{cycle} = \frac{4c}{3}$.

So for any non-zero c, trajectories will loop around in finite time, hence the system is <u>non-Zeno</u>.