Problem 1: Lyapunov's Indirect Method

$$\dot{x}_1 = -x_1 + a x_2 - b x_1 x_2 + x_2^2$$
$$\dot{x}_2 = -(a+b)x_1 + b x_1^2 - x_1 x_2$$

Part 1 :

$$\begin{cases} \dot{x}_1 = 0 \Rightarrow -x_1 + a x_2 - b x_1 x_2 + x_2^2 = 0 & (*) \\ \dot{x}_2 = 0 \Rightarrow -(a+b)x_1 + b x_1^2 - x_1 x_2 = 0 \Rightarrow x_1\left(b x_1 - x_2 - (a+b)\right) = 0 & (**) \end{cases}$$

$$\Rightarrow \begin{cases} x_1 = 0 & \mathrm{I} \\ b x_1 - x_2 - (a+b) = 0 & \mathrm{II} \Rightarrow x_2 = b x_1 - (a+b) \end{cases}$$

$$\mathrm{I}\ \&\ (*) \Rightarrow a x_2 + x_2^2 = 0 \Rightarrow x_2(x_2 + a) = 0 \Rightarrow \begin{cases} x_2 = 0 \\ x_2 = -a \end{cases}$$

$$\mathrm{II}\ \&\ (*) \Rightarrow -x_1 + a(b x_1 - (a+b)) - b x_1 (b x_1 - (a+b)) + (b x_1 - (a+b))^2 = 0$$
$$\Rightarrow -x_1 + \cancel{ab x_1} - \cancel{a^2} - ab - \cancel{b^2 x_1^2} + \cancel{ab x_1} + b^2 x_1 + \cancel{b^2 x_1^2} - \cancel{2ab x_1} - 2b^2 x_1$$
$$+ \cancel{a^2} + 2ab + b^2 = 0$$
$$\Rightarrow -x_1(1 + b^2) + b(a+b) = 0$$

$$\Rightarrow x_1 = \frac{b(a+b)}{1+b^2}$$

$$\Rightarrow x_2 = b x_1 - (a+b) = \frac{b^2(a+b)}{1+b^2} - \frac{(1+b^2)(a+b)}{1+b^2} = -\frac{(a+b)}{1+b^2}$$

Equilibria Points :

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -a \end{bmatrix}, \begin{bmatrix} \frac{b(a+b)}{1+b^2} \\ -\frac{(a+b)}{1+b^2} \end{bmatrix}$$

Part 2 : Linearization

$$A = \left.\frac{\partial f}{\partial x}\right|_{x = \left[\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right]} = \begin{bmatrix} -1 - b x_2 & a - b x_1 + 2x_2 \\ -(a+b) - x_2 + 2b x_1 & -x_1 \end{bmatrix}$$

$$A\Big|_{X=\begin{bmatrix}0\\0\end{bmatrix}} = \begin{bmatrix}-1 & a\\ -(a+b) & 0\end{bmatrix}$$

$$\det(A-\lambda I) = \det\left(\begin{bmatrix}-1-\lambda & a\\ -(a+b) & -\lambda\end{bmatrix}\right) = \lambda(\lambda+1) + a(a+b) = 0$$

$$\Rightarrow \lambda^2 + \lambda + a(a+b) = 0 \Rightarrow \lambda_{1,2} = \frac{-1 \pm \sqrt{1-4a(a+b)}}{2}$$

(I) $1-4a(a+b) < 0 \longrightarrow$ Eigenvalues are complex

Since $Re\{\lambda_{1,2}\} < 0 \Rightarrow$ Locally Asymptotically Stable

(II) $1-4a(a+b) > 0$

$\longrightarrow 1-4a(a+b) < 1$

$\Rightarrow 4a(a+b) > 0 \Rightarrow a+b > 0 \Rightarrow b > -a$ $\longrightarrow$ Both eigenvalues are negative

$\Rightarrow$ Locally Asymptotically Stable

$\longrightarrow 1-4a(a+b) > 1$

$\Rightarrow 4a(a+b) < 0 \Rightarrow a+b < 0 \Rightarrow b < -a$ $\longrightarrow$ One eigenvalue is positive

$\Rightarrow$ Locally Unstable (Saddle)

$\longrightarrow 1-4a(a+b) = 1$ $\longrightarrow$ One eigenvalue equals to zero

$\Rightarrow \underbrace{4a}_{\text{can't be zero } a>0}(a+b) = 0 \Rightarrow b = -a$, $b < 0$

$\Rightarrow$ No claim can be made

(III) $1-4a(a+b) = 0 \longrightarrow \lambda_1 = \lambda_2 = -\frac{1}{2}$

$\Rightarrow 4a(a+b) = 1 \Rightarrow a+b = \frac{1}{4a} \Rightarrow b = \frac{1-4a^2}{4a}$ $\Rightarrow$ Locally Asymptotically Stable

$$A\Big|_{X=\begin{bmatrix}0\\-a\end{bmatrix}} = \begin{bmatrix}-1+ab & -a\\ -b & 0\end{bmatrix}$$

$$\det(A-\lambda I) = \begin{vmatrix}-1+ab-\lambda & -a\\ -b & -\lambda\end{vmatrix} = \lambda(\lambda - ab + 1) - ab = 0$$

$$\Rightarrow \lambda^2 + (1-ab)\lambda - ab = 0 \Rightarrow \lambda_{1,2} = \frac{(ab-1) \pm \sqrt{(1-ab)^2 + 4ab}}{2} = \frac{(ab-1) \pm \sqrt{(1+ab)^2}}{2}$$

$$\Rightarrow \lambda_{1,2} = \frac{(ab-1) \pm (1+ab)}{2} \Rightarrow \begin{cases}\lambda_1 = ab\\ \lambda_2 = -1\end{cases}$$

One eigenvalue is negative. Therefore, the type of stability depends on the other eigenvalue.

(I) $\lambda_1 = ab > 0 \longrightarrow$ One eigenvalue is positive

Since $\xrightarrow{a>0}$ $b > 0$ $\Rightarrow$ Locally Unstable (Improper Unstable Node)

(II) $\lambda_1 = ab < 0 \longrightarrow$ Both eigenvalues are negative

Since $\xrightarrow{a>0}$ $b < 0$ $\Rightarrow$ Locally Asymptotically Stable

(III) $\lambda_1 = ab = 0 \longrightarrow$ Since $b \neq 0$ and $a > 0$, this case can't occur at all.

$$A\Big|_{x=\begin{bmatrix} \frac{b(a+b)}{1+b^2} \\ -\frac{(a+b)}{1+b^2} \end{bmatrix}} = \begin{bmatrix} -1 + \frac{b(a+b)}{1+b^2} & a - \frac{b^2(a+b)}{1+b^2} - \frac{2(a+b)}{1+b^2} \\ -(a+b) + \frac{(a+b)}{1+b^2} + \frac{2b^2(a+b)}{1+b^2} & -\frac{b(a+b)}{1+b^2} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{-1-b^2+ab+b^2}{1+b^2} & \frac{a(1+b^2)-(a+b)(b^2+2)}{1+b^2} \\ \frac{-(a+b)(1+b^2)+(a+b)+2b^2(a+b)}{1+b^2} & -\frac{b(a+b)}{1+b^2} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{ab-1}{1+b^2} & \frac{-b^3-2b-a}{1+b^2} \\ \frac{b^2(a+b)}{1+b^2} & -\frac{b(a+b)}{1+b^2} \end{bmatrix}$$

$$\det(A-\lambda I) = \det\left(\begin{bmatrix} \frac{ab-1}{1+b^2} - \lambda & \frac{-b^3-2b-a}{1+b^2} \\ \frac{b^2(a+b)}{1+b^2} & -\frac{b(a+b)}{1+b^2} - \lambda \end{bmatrix}\right)$$

$$= \left(\lambda - \frac{ab-1}{1+b^2}\right)\left(\lambda + \frac{b(a+b)}{1+b^2}\right) + \left(\frac{b^2(a+b)}{1+b^2}\right)\left(\frac{b^3+2b+a}{1+b^2}\right)$$

$$= \lambda^2 + \left(\frac{ab+b^2-ab+1}{1+b^2}\right)\lambda - \frac{b(ab-1)(a+b)}{(1+b^2)^2} + \frac{b^2(a+b)(b^3+2b+a)}{(1+b^2)^2}$$

$$= \lambda^2 + \lambda + \frac{b(a+b)(\cancel{-ab}+1+b^4+2b^2+\cancel{ab})}{(1+b^2)^2} = \lambda^2 + \lambda + \frac{b(a+b)\cancel{(b^2+1)^2}}{\cancel{(b^2+1)^2}} = 0$$

$$\Rightarrow \lambda^2 + \lambda + b(a+b) = 0 \quad \Rightarrow \lambda_{1,2} = \frac{-1 \pm \sqrt{1-4b(a+b)}}{2}$$

Ⅰ $1-4b(a+b) > 0$

$\Rightarrow 4b(a+b) < 1$

$\longrightarrow 1-4b(a+b) < 1$
$\Rightarrow 4b(a+b) > 0$
$\overset{\text{I}}{\Rightarrow} 0 < 4b(a+b) < 1$ $\longrightarrow$ Both eigenvalues are negative
Locally Asymptotically Stable

$\longrightarrow 1-4b(a+b) > 1$
$\Rightarrow 4b(a+b) < 0$ $\longrightarrow$ One eigenvalue is positive
Locally Unstable (Saddle Node)

$\longrightarrow 1-4b(a+b) = 1$
$4b(a+b) = 0$
$\overset{b \neq 0}{\Rightarrow} b = -a < 0$ $\longrightarrow$ One eigenvalue equals to zero
No claim can be made

Ⅱ $1-4b(a+b) < 0$ $\longrightarrow$ Eigenvalues will be complex conjugate
Since real parts of eigenvalues are negative
$\Rightarrow$ Locally Asymptotically Stable

Ⅲ $1-4b(a+b) = 0$
$\Rightarrow 4b(a+b) = 1$ $\longrightarrow$ $\lambda_1 = \lambda_2 = -\frac{1}{2}$ Both eigenvalues are negative and equal
$\Rightarrow$ Locally Asymptotically Stable

Part 3 :

(a) $a=1$, $b=1$ According to the previous part we will have:

E.P. 1 $x = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \lambda_{1,2} = \frac{-1 \pm \sqrt{1-4\times 2}}{2} = \frac{-1 \pm j\sqrt{7}}{2}$ $Re\{\lambda_{1,2}\} < 0 \Rightarrow$ Locally Asymptotically Stable Focus

E.P. 2 $x = \begin{bmatrix} 0 \\ -1 \end{bmatrix} \Rightarrow \lambda_{1,2} = \pm 1 \Rightarrow$ One eigenvalue is positive $\Rightarrow$ Locally Unstable (Saddle Node)

E.P. 3 $x = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \Rightarrow \lambda_{1,2} = \frac{-1 \pm \sqrt{1-4\times 2}}{2} = \frac{-1 \pm j\sqrt{7}}{2}$ $Re\{\lambda_{1,2}\} < 0 \Rightarrow$ Locally Asymptotically Stable Focus

The phase-plane graph is on the next page.

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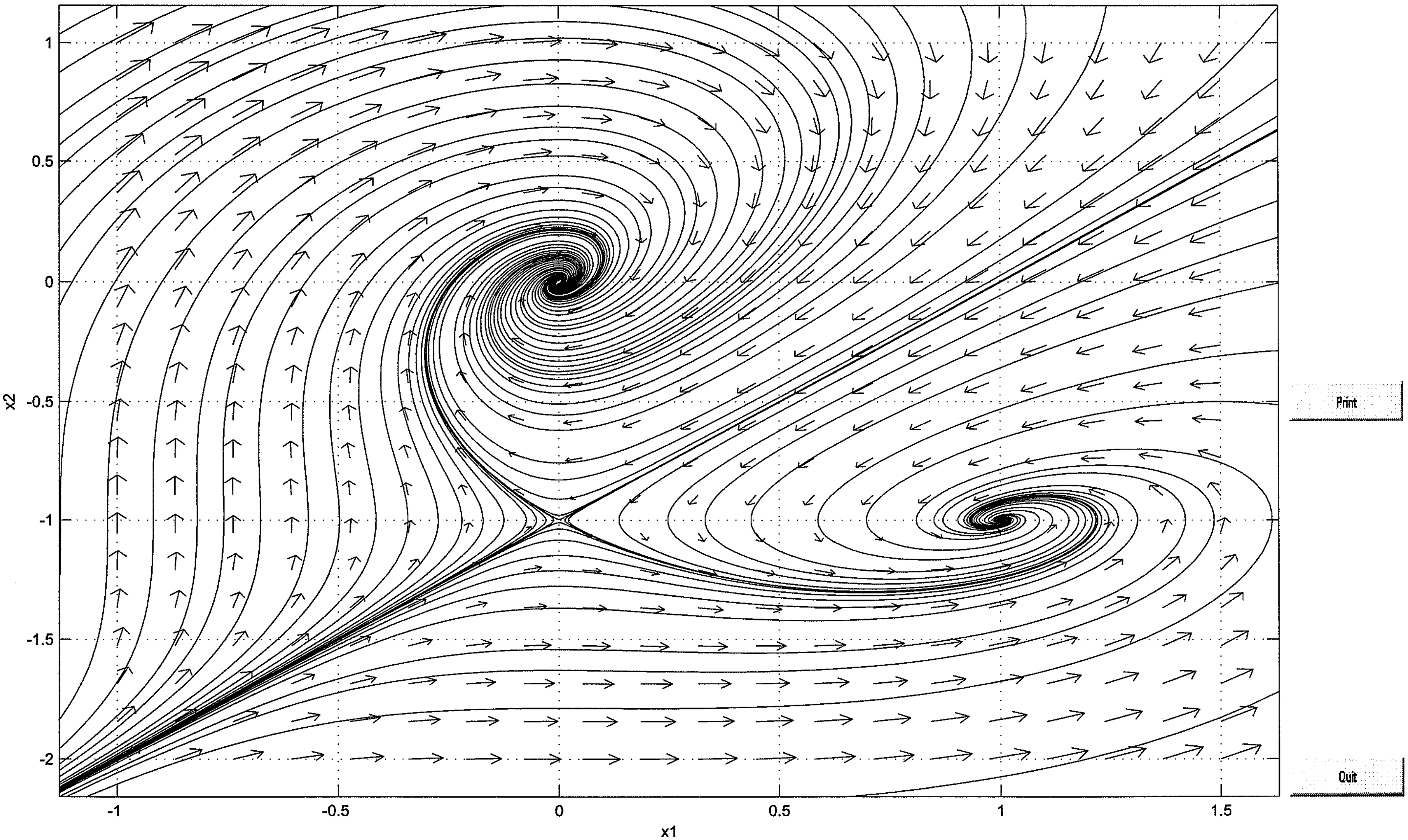
x1 ' = - x1 + a x2 - b x1 x2 + x2^2
x2 ' = - (a + b) x1 + b x1^2 - x1 x2

a = 1
b = 1



Computing the field elements.
Ready. Cursor position: (1.07, -2.53)
Preparing to print the PPLANE6 Display Window. Please be patient.
Printing the PPLANE6 Display Window.
Ready.

As you can see on the phase-plane graph, the two equilibrium points $\begin{bmatrix}0\\0\end{bmatrix}$, $\begin{bmatrix}1\\-1\end{bmatrix}$ are Asymptotically Stable Focus, However, the equilibrium point at $\begin{bmatrix}0\\-1\end{bmatrix}$ is not stable. Comparing with the analytic solutions for λ, we see that results are consistent with each other.

(b) $a=1,\ b=-0.5$ According to the previous part we will have :

E.P. ① : $x = \begin{bmatrix}0\\0\end{bmatrix} \Rightarrow \lambda_{1,2} = \dfrac{-1 \pm \sqrt{1-4\times 0.5}}{2} = \dfrac{-1 \pm j}{2} \Rightarrow$ Locally Asymptotically Stable Focus

E.P. ② : $x = \begin{bmatrix}0\\-1\end{bmatrix} \Rightarrow \begin{cases}\lambda_1 = -0.5\\ \lambda_2 = -1\end{cases} \Rightarrow$ Locally Asymptotically Stable Focus

E.P. ③ : $x = \begin{bmatrix}-0.2\\-0.4\end{bmatrix} \Rightarrow \lambda_{1,2} = \dfrac{-1 \pm \sqrt{1+2\times 0.5}}{2} = \dfrac{-1 \pm \sqrt{2}}{2} \longrightarrow$ One eigenvalue is positive

$\Rightarrow$ Locally Unstable (Saddle Unstable Node)

The phase-plane graph is on the next page.

As you can see on the phase-plane graph, the two equilibrium points $\begin{bmatrix}0\\0\end{bmatrix}$, $\begin{bmatrix}0\\-1\end{bmatrix}$ are Asymptotically Stable Focus. However, the equilibrium point at $\begin{bmatrix}-0.2\\-0.4\end{bmatrix}$ is not stable. Comparing with the analytic solutions for λ, we see that results are consistent with each other.

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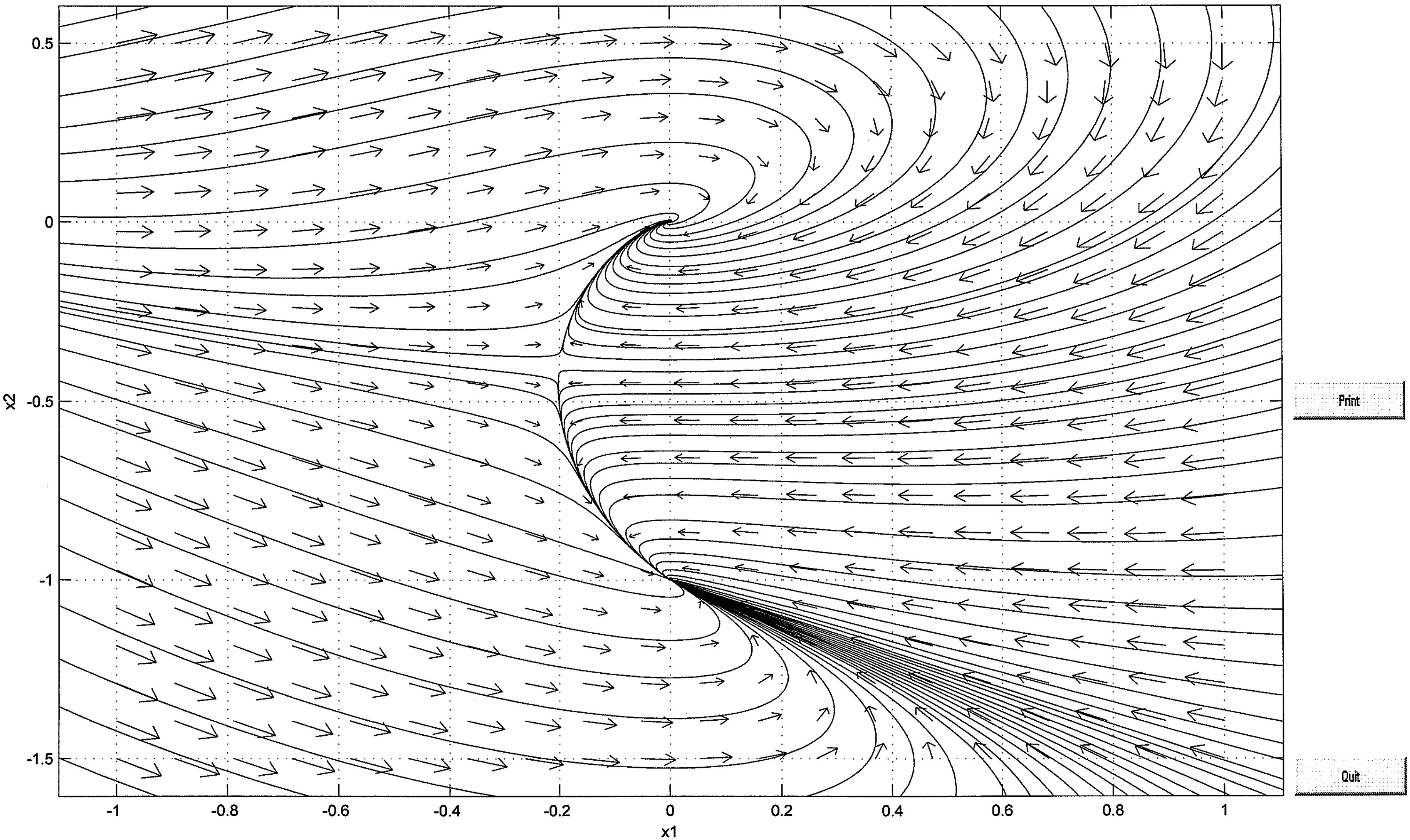
$x1' = -x1 + a\,x2 - b\,x1\,x2 + x2^2$
$x2' = -(a+b)\,x1 + b\,x1^2 - x1\,x2$

a = 1
b = -.5



Computing the field elements.
Ready. Cursor position: (-0.367, 0.175)
The forward orbit from (-0.23, -0.43) --> a possible eq. pt. near (-8.6e-018, -1).
The backward orbit from (-0.23, -0.43) left the computation window.
Ready.

(C) $a = 1, \; b = -2$ According to the previous part's result :

E.P. ① : $x = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \lambda_{1,2} = \dfrac{-1 \pm \sqrt{1-(4)(-1)}}{2} = \dfrac{-1 \pm \sqrt{5}}{2} \rightarrow$ One eigenvalue is positive

$\Rightarrow$ Locally Unstable (Saddle Unstable Node)

E.P. ② : $x = \begin{bmatrix} 0 \\ -1 \end{bmatrix} \Rightarrow \lambda_{1,2} = \begin{cases} -1 \\ -2 \end{cases} \rightarrow$ Locally Asymptotically Stable Focus

E.P. ③ : $x = \begin{bmatrix} 0.4 \\ 0.2 \end{bmatrix} \Rightarrow \lambda_{1,2} = \dfrac{-1 \pm \sqrt{1 - 4(-2)(-1)}}{2} = \dfrac{-1 \pm \sqrt{1-8}}{2} = \dfrac{-1 \pm j\sqrt{7}}{2} \Rightarrow$ Locally Asymptotically Stable Focus

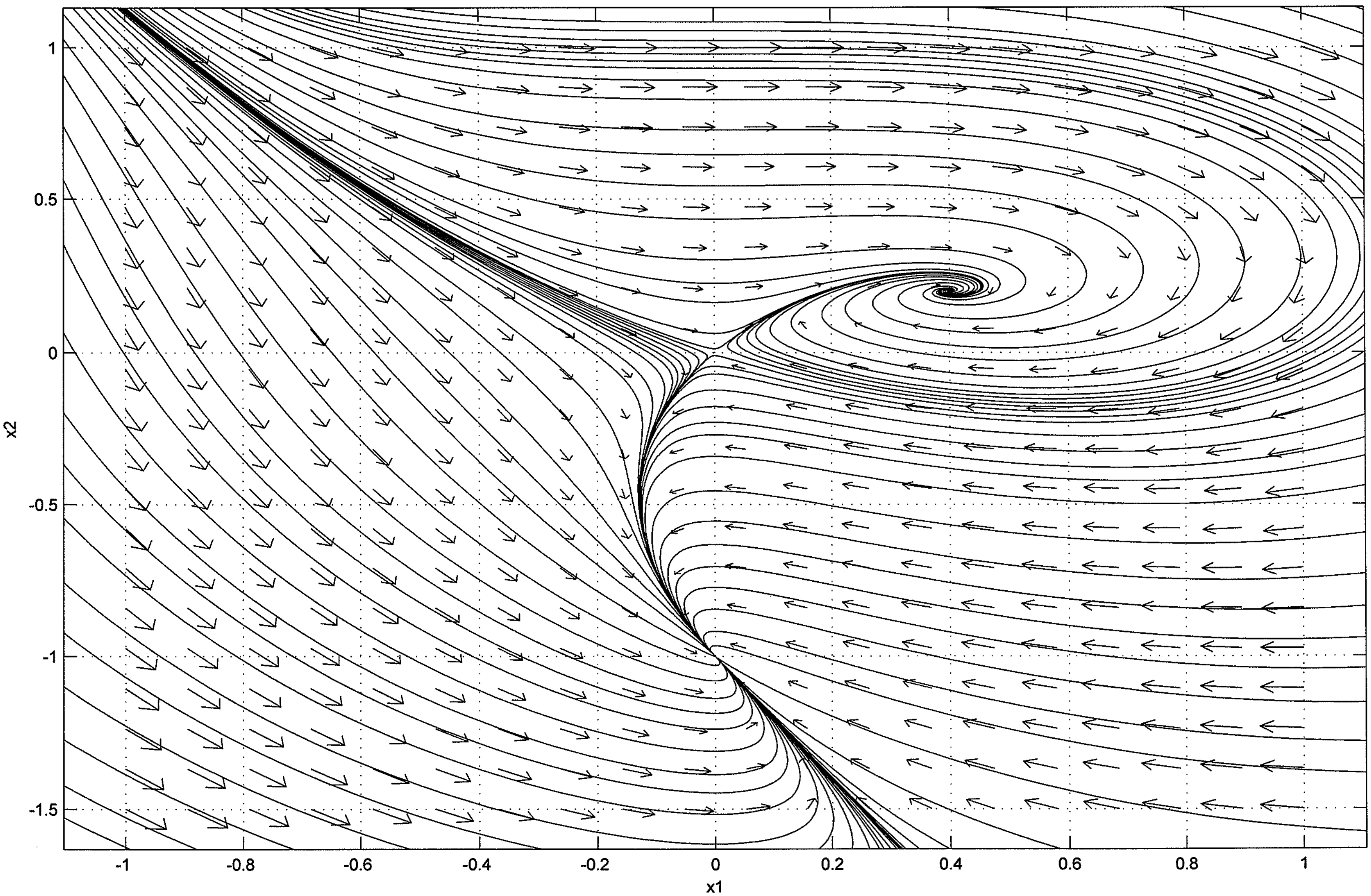
As you can see in the phase-plane graph appearing on the next page, the two equilibrium points $\begin{bmatrix} 0 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} 0.4 \\ 0.2 \end{bmatrix}$ are Asymptotically Stable Focus. However, the equilibrium point at $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is not stable.

Comparing to the analytic solution for λ, we see that results are consistent with each other.

b/

$x1' = -x1 + a\,x2 - b\,x1\,x2 + x2^2$
$x2' = -(a + b)\,x1 + b\,x1^2 - x1\,x2$

$a = 1$
$b = -2$



Print

Quit

The backward orbit from (-0.12, -0.088) left the computation window.
Ready. Cursor position: (-0.474, 0.712)
The forward orbit from (0.057, 0.052) --> a possible eq. pt. near (0.4, 0.2).
The backward orbit from (0.057, 0.052) left the computation window.
Ready.

Problem 2: Longitudinal Aircraft Dynamics

Part ① :

$$\det(A-\lambda I) = \det\left(\begin{bmatrix} -0.313-\lambda & 56.7 & 0 \\ -0.0139 & -0.426-\lambda & 0 \\ 0 & 56.7 & -\lambda \end{bmatrix}\right)$$

$$= -(\lambda+0.313)\lambda(\lambda+0.426) - \lambda\times 56.7\times 0.0139 = -\lambda^3 - (0.313+0.426)\lambda^2 - 0.313\times 0.426\,\lambda - 56.7\times 0.0139\,\lambda$$

$$= -\lambda(\lambda^2 + 0.739\lambda + 0.9214) = 0$$

$$\Rightarrow \begin{cases} \lambda_1 = 0 \\ \lambda_{2,3} = \dfrac{-0.739 \pm \sqrt{(0.739)^2 - 4\times 0.9214}}{2} = \dfrac{-0.739 \pm 1.7719\,j}{2} = -0.3695 \pm 0.8860\,j \end{cases}$$

$$\Rightarrow J = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \qquad \dot{z} = Jz$$

Since one eigenvalue equals to zero, the system would be stable in the sense of Lyapunov. (Since other eigenvalues are complex and have negative real parts)
Since one eigenvalue equals to zero, in the phase plane diagram (in 3D space) we would always be in a plane (according to the initial conditions) and we wouldn't have any vectors in the direction of one of the 3rd dimension..

Part ②: According to the reverse of Lyapunov's stability theorem (the Lyapunov's Theory is both sufficient and necessary for linear systems) since we have a positive definite P and since the system is stable in the sense of Lyapunov and Q must be positive semi-definite $Q \geq 0$.

```
% EECE 571M : Introduction to Hybrid Systems
% Assignment No. 2
% Problem #2: part 3
%
% Omid Namvar Gharehshiran
% Student No.: 31729072

clc;
clear all;
close all;

A=[-0.313,56.7,0;-0.0139,-0.426,0;0,56.7,0];

setlmis([]);

P=lmivar(1,[3 1]);

lmiterm([1 1 1 P],A',1,'s');                    % LMI #1: A1'*P+P*A1

lmiterm([-2 1 1 P],1,1);                         % LMI #3: P

lmisys=getlmis;

% solve the LMI.  Answer is good if tmin <= 0.

[tmin,pfeas]=feasp(lmisys);
Pfinal = dec2mat(lmisys,pfeas,P)
Q=-(A'*Pfinal+Pfinal*A)
eig(Q)
```

Part ③ : In this part we have used a full block matrix as p.

The results of the simulation is on the next page. As you can see, one of the eigenvalues of Q is a very small negative number. We can assume it is equal to zero, since the iterations of the recursive algorithm for finding Pfeas goes on until the considered precision achieved. However, if the iterations goes on, we can see that it goes closer and closer to zero. Therefore, we can consider Q as positive semi-definite since other eigenvalues are positive. This result is also consistent with the answer to part ② of this problem.

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```
 Solver for LMI feasibility problems L(x) < R(x)
    This solver minimizes  t  subject to  L(x) < R(x) + t*I
    The best value of t should be negative for feasibility

 Iteration   :    Best value of t so far

     1                        0.073279
     2                        0.025312
     3                        0.025312
     4                    1.290968e-003
     5                    2.972809e-004
     6                    2.972809e-004
     7                    2.972809e-004
     8                    8.491517e-005
     9                    8.491517e-005
    10                    1.983252e-005
    11                    1.983252e-005
    12                    4.992525e-006
    13                    4.992525e-006
    14                    2.493676e-006
    15                    7.170626e-007
    16                    7.170626e-007
    17                    3.145408e-007
    18                    2.945717e-008
    19                    2.945717e-008
    20                    2.945717e-008
    21                    3.200001e-009
    22                    3.200001e-009
    23                    3.200001e-009
    24                    7.272902e-011
    25                    7.272902e-011
    26                    7.272902e-011
    27                    7.272902e-011
    28                    2.037467e-011
    29                    2.037467e-011
    30                    4.764871e-012
* switching to QR
    31                    4.764871e-012
***                 new lower bound: -1.407375e-010
    32                    6.621230e-013
    33                    6.621230e-013
***                 new lower bound: -2.526742e-011

 Result:  best value of t: 6.621230e-013
          f-radius saturation:  64.219% of R = 1.00e+009

 Marginal infeasibility: these LMI constraints may be
          feasible but are not strictly feasible


Pfinal =
```

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```
   1.0e+008 *

    0.0041   -0.0266   -0.0024
   -0.0266    6.4216    0.0550
   -0.0024    0.0550    0.0029


Q =

   1.0e+008 *

    0.0018   -0.0260    0.0000
   -0.0260    2.2541   -0.0000
    0.0000   -0.0000         0


ans =

   1.0e+008 *

   -0.0000
    0.0015
    2.2544

>>
```

$\boxed{3}$

1. $A = \begin{bmatrix} -1 & \alpha \\ \alpha & -2 \end{bmatrix}$ $\qquad P = \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix}$

$A^T P + PA = -Q$, choose $Q = I \cdot 2$

$$-2I = \begin{bmatrix} -1 & \alpha \\ \alpha & -2 \end{bmatrix} \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} + \begin{bmatrix} -1 & \alpha \\ \alpha & -2 \end{bmatrix} \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix}$$

$$-I = \begin{bmatrix} -p_1 + \alpha p_2 & -p_2 + \alpha p_3 \\ \alpha p_1 - 2p_2 & \alpha p_2 - 2p_3 \end{bmatrix}.$$

3 eqns in 3 unknowns:

$$-1 = -p_1 + \alpha p_2$$
$$0 = \quad - p_2 + \alpha p_3$$
$$-1 \quad \alpha p_2 - 2p_3$$

$$\begin{bmatrix} -1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 & \alpha & 0 \\ 0 & -1 & \alpha \\ 0 & \alpha & -2 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix}$$

$p_2 = \alpha p_3$ and $\alpha p_2 - 2p_3 = -1$

$\alpha^2 p_3 - 2p_3 = -1$

$p_3(\alpha^2 - 2) = -1$

$p_3 = \dfrac{-1}{\alpha^2 - 2} = \dfrac{1}{2 - \alpha^2} > 0$ for $|\alpha| < 1$

$\therefore p_2 = \dfrac{\alpha}{2 - \alpha^2}$

$$-1 = -p_1 + \alpha p_2$$

$$p_1 = 1 + \alpha p_2 = 1 + \frac{\alpha^2}{2-\alpha^2}$$

$$= \frac{2-\alpha^2+\alpha^2}{2-\alpha^2}$$

$$= \frac{2}{2-\alpha^2} > 0 \quad \text{for } |\alpha|<1$$

$$P = \begin{bmatrix} \frac{2}{2-\alpha^2} & \frac{\alpha}{2-\alpha^2} \\ \frac{\alpha}{2-\alpha^2} & \frac{1}{2-\alpha^2} \end{bmatrix} = \frac{1}{2-\alpha^2}\begin{bmatrix} 2 & \alpha \\ \alpha & 1 \end{bmatrix}$$

$$0 = |\lambda I - P| = \frac{1}{2-\alpha^2}\left((\lambda-2)(\lambda-1) - \alpha^2 \right)$$

$$0 = \lambda^2 - 3\lambda + (2-\alpha^2)$$

$$\lambda = \frac{+3 \pm \sqrt{3^2 - 4(2-\alpha)}}{2}$$

and since $1 \le 2-\alpha \le 3$ since $|\alpha(t)| \le 1$,

$$3^2-4(3) \le 3^2-4(2-\alpha) \le 3^2-4(1)$$

$$-3 \le 3^2-4(2-\alpha) \le 5$$

$\therefore$ P has 2 eigenvalues w/ positive real part

$P = P^T \in \mathbb{R}^{2\times 2}$

$\therefore$ $P > 0$ + $V(x) = x^T P x$ is a Lyapunov fcn.

$\underline{\underline{2}}$. $A(t) = \begin{bmatrix} 0 & 1 \\ -1 & -\alpha(t) \end{bmatrix} \qquad \alpha \geq 2.$

$-Q = A^T P + PA$, say $Q = I \cdot 2$

$$-2I = \begin{bmatrix} 0 & -1 \\ +1 & -\alpha \end{bmatrix}\begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} + \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & -\alpha \end{bmatrix}$$

$$= \begin{bmatrix} -p_2 & -p_3 \\ p-\alpha p_2 & p_2-\alpha p_3 \end{bmatrix} + \begin{bmatrix} -p_2 & p_1-\alpha p_2 \\ -p_3 & p_2-\alpha p_3 \end{bmatrix}$$

$$-2I = \begin{bmatrix} -2p_2 & p_1-\alpha p_2-p_3 \\ p_1-\alpha p_2-\alpha p_3 & 2(p_2-\alpha p_3) \end{bmatrix}$$

$\therefore \quad -1 = -p_2 \qquad \Longrightarrow p_2 = 1$

$0 = p_1 - \alpha p_2 - p_3$

$-1 = p_2 - \alpha p_3$

$\Rightarrow \quad p_3 = \dfrac{p_2+1}{\alpha} = \dfrac{2}{\alpha} \quad > 1$ for $\alpha \geq 2.$

$p_1 = \alpha p_2 + p_3 = \dfrac{2}{\alpha}.$

$P = \begin{bmatrix} 2/\alpha & 0 \\ 0 & 2/\alpha \end{bmatrix} > 0$ w/ eigenvalues $\lambda_{1,2} = \dfrac{2}{\alpha}$.

3. $A = \begin{bmatrix} -1 & 0 \\ \alpha & -2 \end{bmatrix}$, $P = \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix}$

$$-Q = A^T P + PA, \quad Q = \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & \alpha \\ 0 & -2 \end{bmatrix} \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} + \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ \alpha & -2 \end{bmatrix}$$

$$= \begin{bmatrix} -p_1 + \alpha p_2 & -p_2 + \alpha p_3 \\ -2p_2 & -2p_3 \end{bmatrix} + \begin{bmatrix} -p_1 + \alpha p_2 & -2p_2 \\ -p_2 + \alpha p_3 & -2p_3 \end{bmatrix}$$

$$\begin{bmatrix} -2 & 0 \\ 0 & -4 \end{bmatrix} = \begin{bmatrix} 2(-p_1 + \alpha p_2) & -3p_2 + \alpha p_3 \\ -3p_2 + \alpha p_3 & -4p_3 \end{bmatrix}$$

$$\Rightarrow -2 = (-p_1 + \alpha p_2) \cdot 2$$
$$0 = -3p_2 + \alpha p_3$$
$$-4 = -4 \cdot p_3$$

$\Rightarrow p_3 = 1$

$\Rightarrow 3p_2 = \alpha p_3$

$$p_2 = \frac{\alpha}{3} \cdot 1 = \frac{\alpha}{3}$$

$\Rightarrow -1 = -p_1 + \alpha p_2 \quad \therefore \quad p_1 = \alpha p_2 + 1$

$$P = \begin{bmatrix} \frac{\alpha^2}{3} + 1 & \frac{\alpha}{3} \\ \frac{\alpha}{3} & 1 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} \alpha^2 + 3 & \alpha \\ \alpha & 3 \end{bmatrix}$$

$$0 = |\lambda I - P| = \frac{1}{3}\left((\lambda - \alpha^2 - 3)(\lambda - 3) - \alpha^2\right)$$
$$= \lambda^2 - 3\lambda - \alpha^2\lambda - 3\lambda + 9 + 3\alpha^2 - \alpha^2$$
$$= \lambda^2 - (6 + \alpha^2)\lambda + (9 + 2\alpha^2)$$

$$\lambda = \frac{+(6+\alpha^2) \pm \sqrt{(6+\alpha^2)^2 - 4(9+2\alpha^2)}}{2}$$

$$= 3 + \frac{\alpha^2}{2} \pm \sqrt{36 + 12\alpha^2 + \alpha^4 - 36 - 8\alpha^2} \cdot \frac{1}{2}$$

$$= 3 + \frac{\alpha^2}{2} \pm \sqrt{\frac{\alpha^4}{4} + 1}$$

So need to show

$$\frac{\alpha^4}{4} + 1 < \left(3 + \frac{\alpha^2}{2}\right)^2$$

$$\frac{\alpha^4}{4} + 1 < 9 + 3\alpha^2 + \frac{\alpha^4}{4}$$

$$0 < 8 + 2\alpha^2$$ holds for any α.

$\therefore$ λ will have positive roots and since $P = P^T \in \mathbb{R}^{2\times 2}$

$\therefore$ $P > 0$.

$\boxed{14}$

$$\dot{x}_1 = x_2(1-x_1^2)$$
$$\dot{x}_2 = -(x_1+x_2)(1-x_1^2)$$

1. $0 = x_2(1-x_1^2) \qquad = x_2 - x_1^2 x_2$

$0 = -(x_1+x_2)(1-x_1^2) \qquad = -x_1 + x_1^3 - x_2 + x_1^2 x_2$

Equilibria at $x^* = \begin{cases} (+1, a) & a \in \mathbb{R} \\ (-1, a) & a \in \mathbb{R} \\ (0, 0) \end{cases}$

2. Lyapunov's indirect method:

$$\left.\frac{\partial f}{\partial x}\right|_{x=0} = \left.\begin{bmatrix} \partial f_1/\partial x_1 & \partial f_1/\partial x_2 \\ \partial f_2/\partial x_1 & \partial f_2/\partial x_2 \end{bmatrix}\right|_{x_1=0,\, x_2=0}$$

$$= \left.\begin{bmatrix} -2x_1x_2 & -x_1^2 \\ -(1-x_1^2) + 2x_1(x_1+x_2) & -(1-x_1^2) \end{bmatrix}\right|_{x_1=0,\, x_2=0}$$

$$= \begin{bmatrix} 0 & 0 \\ -1 & 1 \end{bmatrix}$$

$\Rightarrow 0 = \lambda(\lambda - 1)$ one eigval @ 0, $\therefore$ can not use indirect method

Lyapunov's direct method:

$V(x) = x^T P x = x_1^2 p_1 + 2x_1x_2p_2 + x_2^2 p_3$

$\dot{V}(x) = x^T P \dot{x} + \dot{x}^T P x$

$= 2x_1 p_1 \cdot \dot{x}_1 + 2x_2 p_2 \cdot \dot{x}_1 + 2x_1 p_2 \cdot \dot{x}_2 + 2x_2 p_3 \dot{x}_2$

$= \dot{x}_1(2p_1x_1 + 2p_2x_2) + \dot{x}_2(2p_2x_1 + 2p_3x_2)$

$$= 2\left(p_1x_1 + p_2x_2\right)\left(x_2(1-x_1^2)\right) + 2(p_2x_1 + p_3x_2)\left(-(x_1+x_2)(1-x_1^2)\right)$$

$$\tfrac{1}{2}\dot{V}(x) = (1-x_1^2)\left[p_1x_1x_2 + p_2x_2^2 - p_2x_1x_2 - p_2x_1^2 - p_3x_1x_2 - p_3x_2^2\right]$$

$$= (1-x_1^2)\left[x_1^2(-p_2) + x_2^2(p_2-p_3) + x_1x_2(p_1-p_2-p_3)\right]$$

$$\dot{V}(x) = 2(1-x_1^2)\cdot x^T \underbrace{\begin{bmatrix} -p_2 & \frac{p_1-p_2-p_3}{2} \\ \frac{p_1-p_2-p_3}{2} & p_2-p_3 \end{bmatrix}}_{-Q} x$$

And for x "near" the origin, $1-x_1^2 > 0$.
$\therefore$ need to find p_1, p_2, p_3 s.t. $P>0$ + $Q>0$.

$\therefore$ $p_1>0$, $p_3>0$ and $-p_2<0$, $p_2-p_3<0$
$\therefore$ $p_2>0$ $p_3-p_2>0$

So can't choose P to be diagonal.

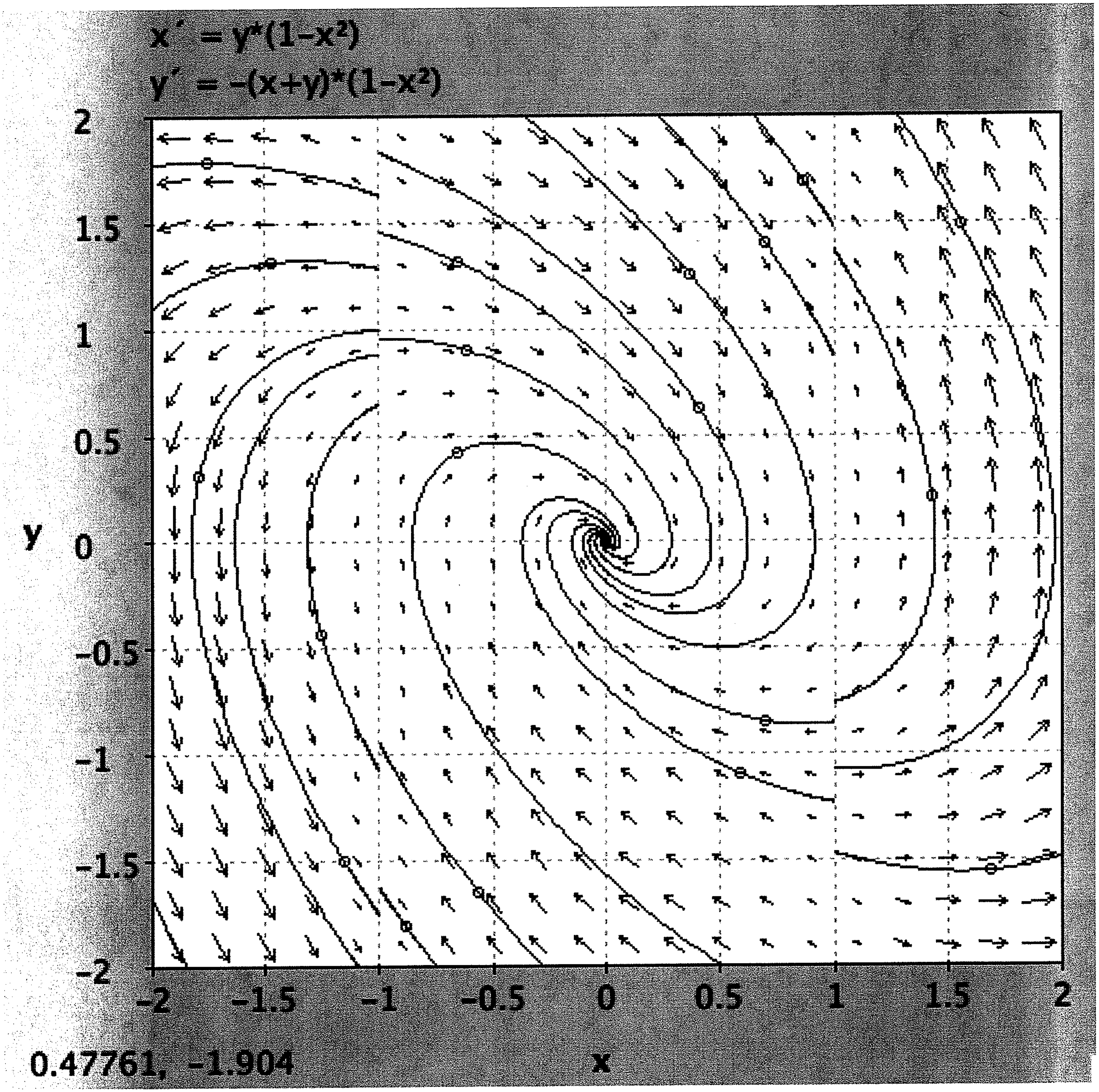
$$P = \begin{bmatrix} 1 & 1/2 \\ 1/2 & 2 \end{bmatrix} \Rightarrow -Q = \begin{bmatrix} -1/2 & -3/4 \\ -3/4 & -3/2 \end{bmatrix}$$

$\lambda_{1/2} = 0.7929,$ 2.2071 $\qquad$ $\lambda_{1,2} = 1.9014,$ $0.0986.$

3. See attached.

4. From phase plane, it seems convergent trajectories will be contained w/in the ellipse passing through $x = \begin{bmatrix} 1^- \\ 0 \end{bmatrix}$.

For $x = \begin{bmatrix} 1^- \\ 0 \end{bmatrix}$, $V(x) = x^TPx = 1^-$.
$\therefore$ the region of attraction is $\{x \mid V(x) = x^TPx < 1\}$.


x´ = y*(1–x²)
y´ = –(x+y)*(1–x²)
2
1.5
1
0.5
0
–0.5
–1
–1.5
–2
y
–2
–1.5
–1
–0.5
0
0.5
1
1.5
2
x
0.47761, –1.904