EECE 571M/491M, Spring 2008

Lecture 2

Modeling Continuous and Discrete Systems

Dr. Meeko Oishi

Electrical and Computer Engineering

University of British Columbia, BC

http://courses.ece.ubc.ca/571m.html

moishi@ece.ubc.ca

Tomlin LN 1, 2

Review

- Course logistics:
  - See the class webpage for syllabus details
  - Midterms, February 15 and March 28 2008 (possibly take-home)
  - Final project, April 18 2008
    - Grad students -- presentation required
    - Undergrad students -- presentation for bonus points

Review

- Introduction
  - Hybrid systems are pervasive
  - Specialized techniques are required for analysis and synthesis of hybrid systems
  - Hybrid systems contain both **continuous** and **discrete** elements
- Switching and stability
  - Hybrid systems with modes with stable dynamics are **not always** stable
  - Hybrid systems with modes with unstable dynamics are **not always** unstable
- Applications
  - Longitudinal and lateral dynamics of aircraft
  - Biological networks, ecological systems
  - Automotive systems (within a car, as well as a hierarchical)
  - Robotics; mechanical systems with discontinuities

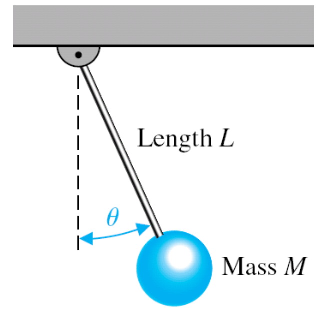
Brief History

- Hybrid systems has been studied rigorously over the past ~15-20 years
- Earlier work in continuous systems with non-smoothness
  - Sliding mode control (Utkin, 1992)
  - Anti-windup filters
  - Filippov (1988), and others…
- Discrete event systems
  - Finite automata theory (1950s)
  - Model checking
  - Timed automata
- "Hybrid" work pushed by need for verification of systems for which timing is critical (and not captured by finite automata)

Today's lecture

- Continuous systems
  - Differential or difference equations
  - Existence, uniqueness
  - Common forms
- Discrete event systems
  - Finite state machines, automata
  - Existence, uniqueness
- Commonalities in continuous and discrete systems
  - Open-loop vs. closed-loop
  - Modeling variations

Continuous systems



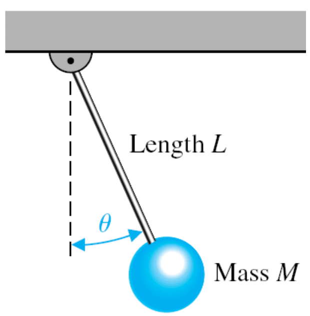
- Differential equations with output

$$\dot{x} = f(x,u)$$
$$y = h(x)$$

  - State *x(t)*
  - Input *u(t)*
  - Output *y(t)*
- In general, *x, u, y* are vectors
- Transform *n*th order differential equations into sets of coupled 1st order differential equations

$$ML\ddot{\theta} = -Mg\sin\theta \quad \Rightarrow \quad x = \begin{bmatrix}\theta \\ \dot{\theta}\end{bmatrix}, \quad \dot{x} = \begin{bmatrix}\dot{\theta} \\ -\frac{g}{L}\sin\theta\end{bmatrix}$$

Continuous systems



- Differential equations with output

$$\dot{x} = f(x,u)$$
$$y = h(x)$$

  - State *x(t)*
  - Input *u(t)*
  - Output *y(t)*
- **Solution** (or **trajectory**) of this ODE is *x(t)*
- Solution requires knowledge of the initial condition $x(0) = x_0$

$$ML\ddot{\theta} = -Mg\sin\theta \quad \Rightarrow \quad x = \begin{bmatrix}\theta \\ \dot{\theta}\end{bmatrix}, \quad \dot{x} = \begin{bmatrix}\dot{\theta} \\ -\frac{g}{L}\sin\theta\end{bmatrix}$$

Continuous systems

- What can go wrong with solving ODEs?
  - *No solutions*
    - Exercise: Show that *dx/dt = -sign(x), x(0)* = 0 does NOT have a solution for t >= 0.
  - Multiple solutions
    - Exercise: Show that for $dx/dt = x^{1/3}$, x(0) = 0, both functions

$$x(t) = \tfrac{2}{3}t^{3/2}$$
$$x(t) = 0$$

are solutions to the differential equation.

- Theorem (Existence and uniqueness of solutions):
  If *f(x)* is **Lipschitz continuous**, then the differential equation dx/dt = f(x), *x*(0) = x_0 has a unique solution x(t) for t >0.
- This ensures **smoothness** by bounding the slope of *f*

Continuous systems

Lipschitz continuity

- *f(x)* is **Lipschitz continuous** if there exists a constant $K > 0$ such that for all *x, y*

$$|| f(x) - f(y) || < K || x - y ||$$

(where || . || indicates the **vector norm**)

- *K* is the **Lipschitz constant**
- Note that
  - If *f* is Lipschitz continuous, it is also continuous.
  - A Lipschitz continuous function is not necessarily differentiable.
  - All differentiable functions with bounded derivatives are Lipschitz

Continuous systems

Vector norms

- Common types of vector norms
  - 1-norm $|| x ||_1 = |x_1| + |x_2| + \ldots + |x_n|$
  - Euclidean norm (2-norm) $|| x ||_2 = (x^T x)^{1/2}$
  - ∞-norm $|| x ||_\infty = \max_i |x_i|$
- These norms are all equivalent in that there exists constants c_1, c_2 such that

$$c_1 || x ||_a \le || x ||_b \le c_2 || x ||_a$$

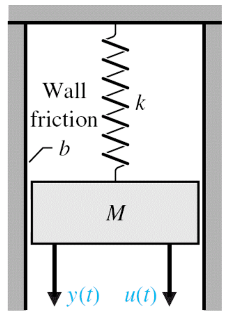
- **Useful norm properties**
  - $|| x || \ge 0$ for $x \ne 0$
  - $|| x + y || \le || x || + || y ||$
  - $|| \alpha x || = | \alpha | \, || x ||$ for scalar α

Continuous systems

- Classified according to the form of f(x,u)
- Autonomous dx/dt = f(x)
  - Linear dx/dt = Ax(t)
  - Affine dx/dt = Ax(t) + b
- Non-autonomous dx/dt = f(x,u)
  - Linear dx/dt = Ax(t) + Bu(t)
  - Affine dx/dt = Ax(t) + Bu(t) + b
  - Nonlinear affine dx/dt = f(x) + g(x)u

Continuous systems

Example: Spring-mass-damper system



- *n*th-order differential equation

$$M\frac{d^2y(t)}{dt^2} + b\frac{dy(t)}{dt} + ky(t) = r(t)$$

- State-space differential equations

$$\dot{x}_1(t) = \dot{y}(t)$$
$$\dot{x}_2(t) = -\frac{k}{M}y(t) - \frac{b}{M}\dot{y}(t) + \frac{1}{M}u(t)$$

with state x = [y dy/dt]′

- This is a **linear non-autonomous system**

Continuous Systems

- Linear systems
  - A function $f(x)$ is ***linear*** if it fulfills the following two properties:

$$\begin{aligned} &1.\ \text{Superposition}: && f(x+y) = f(x) + f(y) \\ &2.\ \text{Scaling}: && f(ax) = a\,f(x) \end{aligned}$$

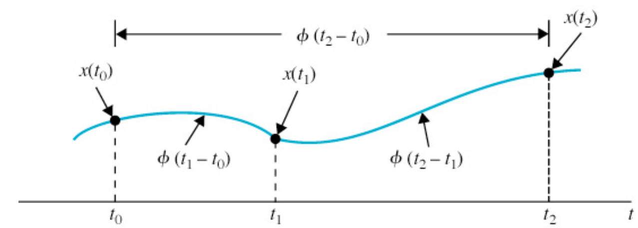
  - Exercise: Show that these two conditions hold for autonomous and non-autonomous linear dynamical systems.

Continuous Systems

- Linear systems
  - Closed-form solution for $\dot{x} = Ax, \quad x(0) = x_0$

$$x(t) = \Phi(t)x(0), \qquad \Phi(t) = e^{At}$$

$$\Phi(t) = e^{At} = I + At + A^2\frac{t^2}{2!} + \ldots + \frac{t^k}{k!}A^k + \ldots$$



Continuous Systems

- Linear systems
  - Closed-form solution for

$$\dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0$$

is

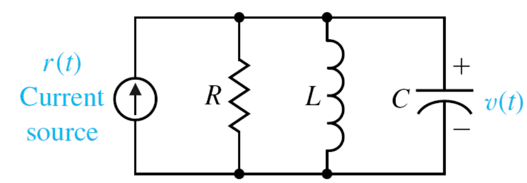
$$x(t) = \underbrace{\Phi(t)x(0)}_{\text{Natural response}} + \underbrace{\int_0^t \Phi(t-\tau)Bu(\tau)d\tau}_{\text{Forced response}}$$

Natural response **Forced response**

- For general dynamical systems, it is very difficult to find closed-form solutions.

Continuous systems

- Another example of a continuous dynamical system

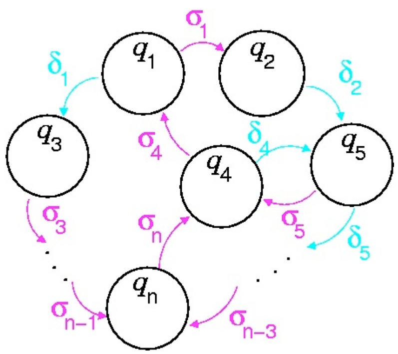


- Kirchoff's current law yields an integro-differential equation

$$C\frac{dv(t)}{dt} + \frac{1}{R}v(t) + \frac{1}{L}\int_0^t v(t) = r(t)$$

that can be placed into standard linear state-space form

Discrete systems

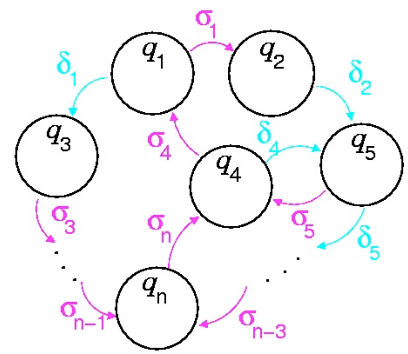


- Finite state machine

  $$G = (Q, \Sigma, R)$$

  - Set of discrete states (or **modes**) Q
  - Set of **events** (or transitions) Σ
  - Transition relation R
  - Initial set of states Q_0

Discrete systems



$$q(k+1) = R(q(k), \sigma(k))$$

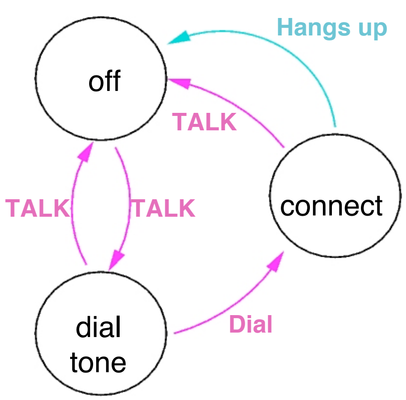
- Finite state machine

  $$G = (Q, \Sigma, R)$$

  - Set of discrete states (or **modes**) Q
  - Set of **events** (or transitions) Σ
  - Transition relation R
  - Initial set of states Q_0

- Solutions are **executions** of the automata
- This is a string of modes and a string of events that trigger those modes.

Discrete Systems



Hand-held phone

- Modes
  $Q = \{\text{off}, \text{dial tone}, \text{connect}\}$
- Events
  Σ = {TALK, Dial, Hangs up}
- Transition function
  As shown graphically

Discrete Systems

- Uniqueness of an execution depends on *R*.
  - G is **deterministic** if and only if $|R(q,s)| \leq 1$ for all possible combinations of q in Q and s in S.
  - In this case, R is called a **transition function**
  - Otherwise G is **non-deterministic**

- Set notation
  - Element of a set
  - Product of multiple sets
  - Power set
  - Language of an automaton

Continuous & discrete systems

- Model variations
  - Control disturbances as well as control inputs
  - Stochastic processes (as continuous inputs or discrete events)
  - Difference equations instead of differential equations in the continuous dynamics
- Additional parameters will be defined to suit particular problem issues.
  - Marked vs unmarked modes in discrete event systems
  - Domains (invariants) and bounded sets in continuous states and inputs
  - Others…

Continuous & discrete systems

- Role of control
  - Continuous control enters through *u* and usually represents physical forces acting on a system
  - Discrete control enters through enabling/disabling of specific events and can represent mode-logic or other structure superimposed on the system
  - With state-based feedback, a non-autonomous system becomes autonomous (control no longer explicitly appears)

Continuous & discrete systems

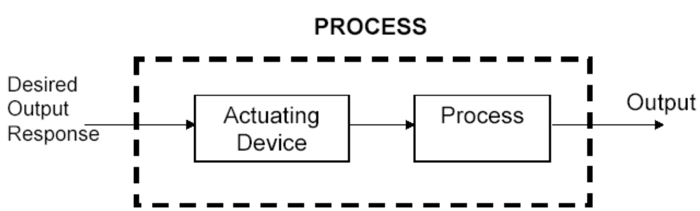


Figure 1.2 Open-loop Control System (without feedback)

An open-loop control system uses an actuating device to control the process directly.

Continuous & discrete systems

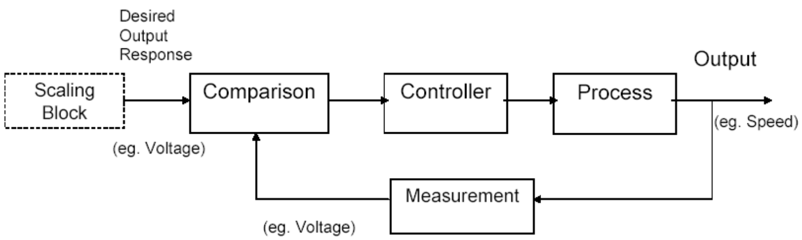
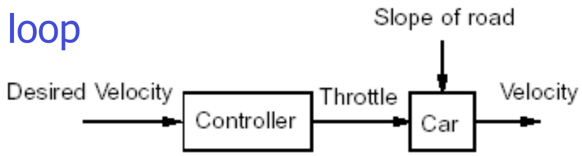


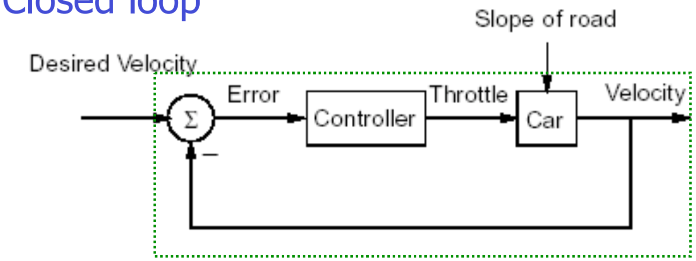
Figure 1.3 Closed-loop Control System (with feedback)

Automobile Cruise Control

- Open loop



- Closed loop



Summary

- Continuous and discrete systems defined by states, inputs, functions/relations, and initial conditions
- Solutions or executions depend on the dynamics as well as the initial condition.
- Special care must be taken for continuous systems to ensure both existence and uniqueness

- Note: If you are not already, become familiar with set notation and norm functions, etc. presented here -- these will be used throughout the course

Hybrid Systems