EECE 571M/491M, Spring 2008
Lecture 6

Hybrid System Stability

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Tomlin LN 6

Review

- (Nonlinear) continuous system stability
  - Indirect method
    - If the linearized system has Re(λ_i)<0 (for all *i*), the nonlinear system is locally asymptotically stable about the equilibrium point
    - If the linearized system has Re(λ_i)>0 (for at least one *i*), the nonlinear system is locally asymptotically unstable about the equilibrium point
    - If the linearized system has Re(λ_i)=0 (for at least one *i*), no claims may be made about the stability of the nonlinear system
  - Direct method
    - If you can find a Lyapunov function, then the system is stable.
    - If you cannot find a Lyapunov function, you cannot claim anything about the stability of the system about the equilibrium point.
    - Lyapunov functions are "energy-like functions"
    - Lypaunov functions are a **sufficient** condition for stability
- Special case: Lyapunov theory for linear systems
  - Necessary and sufficient conditions
  - Quadratic Lyapunov functions

Review: Lyapunov equation

- For the dynamical system

$$\dot{x} = Ax, \quad x(0) = x_0$$

- consider the **quadratic Lyapunov function**

$$V(x) = x^T P x, \quad P = P^T$$

- whose time-derivative

$$\begin{aligned} \dot{V}(x) &= x^T P \dot{x} + \dot{x}^T P x \\ &= x^T P A x + (Ax)^T P x \\ &= x^T (PA + A^T P) x \end{aligned}$$

- can be written as

$$\dot{V} = -x^T Q x, \quad Q \triangleq -(PA + A^T P) = Q^T$$

Review: Lyapunov stability

Theorem: Lyapunov stability for linear systems

- The equilibrium point x*=0 of dx/dt = Ax is asymptotically stable **if and only if** for all matrices Q = Q^T > 0 there exists a matrix P = P^T > 0 such that

$$PA + A^T P + Q = 0$$

- For a given Q, P will be unique
- The solution P is given by $P = \int_0^\infty e^{A^T t} Q e^{At} dt$
- To numerically solve for P, formulate the linear matrix inequality

$$\begin{aligned} A^T P + PA &< 0 \\ P &> 0 \end{aligned}$$

- And invoke the Matlab LMI toolbox

Today's lecture

- Review
  - Continuous system stability
  - Linear quadratic lyapunov theory
  - LMIs
- Introduction to hybrid stability
  - Hybrid equilibrium
  - Hybrid stability
- Multiple Lyapunov functions
  - Hybrid systems
  - Switched systems

Hybrid equilibrium

- Definition:

  The continuous state $x^* \in \mathcal{R}^n$ is an equilibrium point of the autonomous hybrid automaton
  H = (Q, X, f, R, Dom, Init) if
  - $f(q, 0) = 0$ for all $q \in Q$
  - $R(q, 0) \subseteq Q \times 0$
- Assume without loss of generality that x* = 0
- Jumps out of (q,0) are allowed as long as in the new mode, the continuous state is x = 0

Hybrid stability

- Definition:

  The equilibrium point x* = 0 of the autonomous hybrid automaton H = (Q, X, f, R, Dom, Init) is **stable** if for all $\varepsilon > 0$ there exists a $\delta > 0$ such that for all executions (τ, q, x) starting from the state (q_0, x_0),

  $$\|x_o\| < \delta \Rightarrow \|x(t)\| < \epsilon, \quad t \in \tau$$
- As before, the continuous state must merely stay within some arbitrary bound of the equilibrium point -- convergence is not required

Hybrid stability

- Definition:

  The equilibrium point x* = 0 of the autonomous hybrid automaton H = (Q, X, f, R, Dom, Init) is **asymptotically stable** if it is stable and δ can be chosen such that for all executions (τ, q, x) starting from the state (q_0, x_0),

  $$\|x_o\| < \delta \Rightarrow \lim_{t \to \tau_\infty} \|x(t)\| = 0$$
- Note that (τ, q, x) is assumed to be an infinite execution, with $\tau_\infty = \sum_i (\tau_i' - \tau_i)$
- Recall that for non-Zeno executions, $t_\infty = \infty$

Hybrid stability

Controller 1: $dx/dt = A_1x$ — STABLE ⇄ Controller 2: $dx/dt = A_2x$ — STABLE → **UNSTABLE**

Controller 1: $dx/dt = A_1x$ — STABLE ⇄ Controller 2: $dx/dt = A_2x$ — STABLE → **STABLE**

Hybrid stability

Controller 1: $dx/dt = A_1x$ — UNSTABLE ⇄ Controller 2: $dx/dt = A_2x$ — UNSTABLE → **UNSTABLE**

Controller 1: $dx/dt = A_1x$ — UNSTABLE ⇄ Controller 2: $dx/dt = A_2x$ — UNSTABLE → **STABLE**

Hybrid stability: Example 1

$$Q = \{q_1, q_2\}$$

$$f = \{A_1x, A_2x\}, \quad A_1 = \begin{bmatrix} -1 & 10 \\ -100 & -1 \end{bmatrix}, \; A_2 = \begin{bmatrix} -1 & 100 \\ -10 & -1 \end{bmatrix}$$

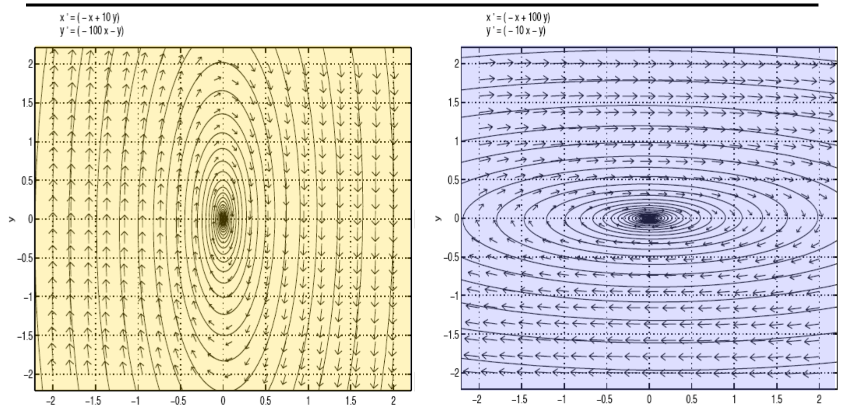
$$Init = Q \times \{X \setminus 0\}$$

$$Dom = \{q_1, x_1x_2 \le 0\} \cup \{q_2, x_1x_2 \ge 0\}$$

$$R(q_1, \{x \mid x_1x_2 \ge 0\}) = (q_2, x), R(q_2, \{x \mid x_1x_2 \le 0\}) = (q_1, x)$$

Controller 1: $dx/dt = A_1x$ — STABLE ⇄ Controller 2: $dx/dt = A_2x$ — STABLE → **UNSTABLE**

Hybrid stability: Example 1



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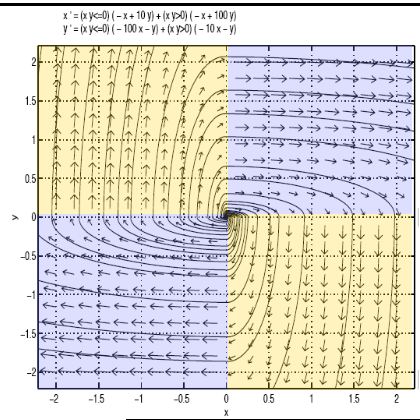
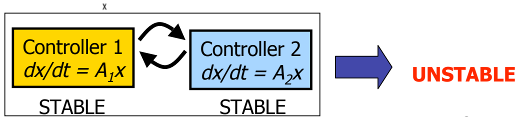
Hybrid stability: Example 1

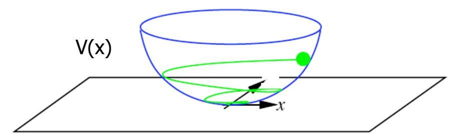


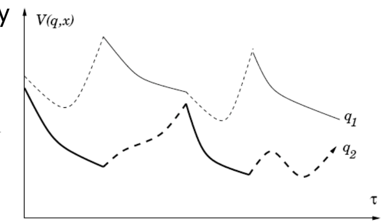


Multiple Lyapunov functions

Consider a Lyapunov-like function V(q,x):

- When the system is evolving in mode q, V(q,x) must decrease or maintain the same value
- Every time mode q is re-visited, the value V(q,x) must be lower than it was last time the system entered mode q.
- When the system switches into a new mode q', V may jump in value
- For inactive modes p, V(p,x) may increase
- Requires solving for V directly





Hybrid Lyapunov stability

Theorem: Lyapunov stability for hybrid systems

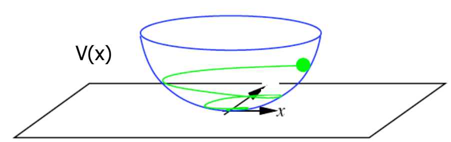
- Consider a hybrid automaton H with equilibrium point x*=0. Assume that there exists an open set $D \subset Q \times \mathcal{R}^n$ such that $(q,0) \in D$ for some $q \in Q$. Let $V : D \to R$ be a continuously differentiable function in x such that for all $q \in Q$:

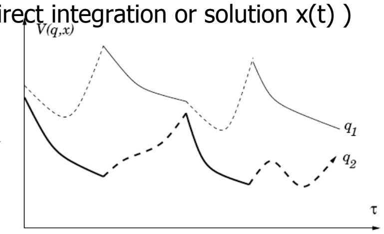
  1. $V(q,0) = 0;$
  2. $V(q,x) > 0$ *for all* x, $(q,x) \in D \backslash \{0\}$, *and*
  3. $\frac{\partial V(q,x)}{\partial x} f(q,x) \leq 0$ *for all* x, $(q,x) \in D$

- If for all (τ,q,x) starting from $(q_0, x_0) \in Init \cap D$, and all $q' \in Q$ the sequence $\{V(q(\tau_i), x(\tau_i)) : q(\tau_i) = q'\}$ is non-increasing (or empty), then x*=0 is a stable equilibrium point of H.

Multiple Lyapunov functions

- Lyapunov-like function in each mode must
  - Decrease when that mode is active
  - Enter that mode with a value lower than the last time the mode was entered
- Valid for any continuous dynamics (including nonlinear) and any reset map (including ones with discontinuities in the state)
- Requires construction of sequence of 'initial' values of V(q,x) each time a mode is re-visited
  (Defeats goal of bypassing direct integration or solution x(t))





Hybrid stability: Example 1

- Candidate Lyapunov function

$$V(q,x) = \begin{cases} x^T P_1 x & \text{if } q = q_1 \\ x^T P_2 x & \text{if } q = q_2 \end{cases}$$

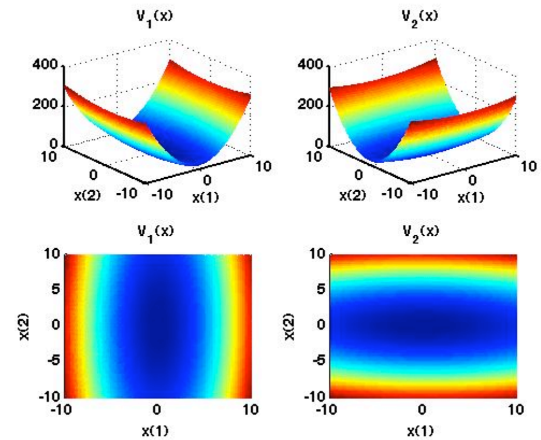
- Check whether the function meets the requirements for stability
  - Lyapunov function in each mode

$$\begin{aligned} \frac{\partial V}{\partial x}(q,x) f(q,x) &= \frac{d}{dt} V(q, x(t)) \\ &= \dot{x}^T P_i x + x^T P_i \dot{x} \\ &= x^T A_i^T P_i x + x^T P_i A_i x \\ &= x^T (A_i^T P_i + P_i A_i) x \\ &= -x^T I x \\ &= -\|x\|^2 \leq 0 \end{aligned}$$

  - Sequence for each mode

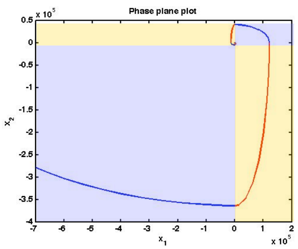
Hybrid stability: Example 1

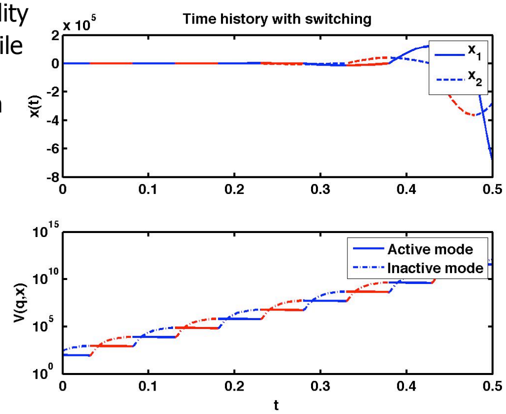
- With $Q_1 = Q_2 = I$
- Lyapunov functions in each mode
- Level sets are ellipses



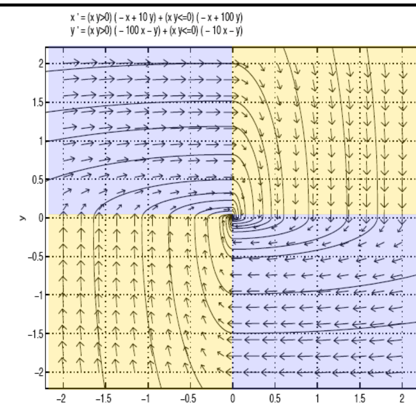
Hybrid stability: Example 1

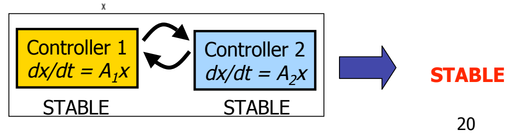
- These Lyapunov-like functions are not acceptable candidates to show hybrid stability
- V(q,x) decreases while mode q is active
- But V is higher when mode is re-visited





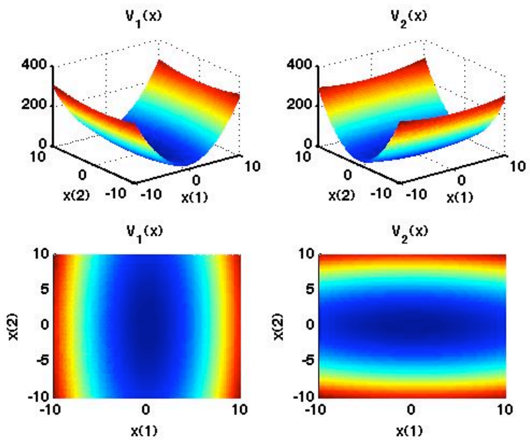
Hybrid stability: Example 2





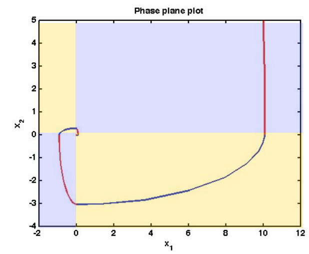
Hybrid stability: Example 2

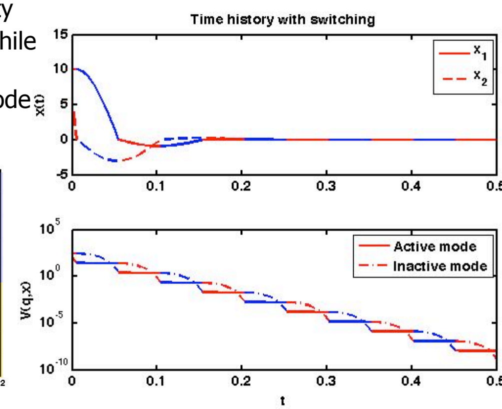
- With $Q_1=Q_2=I$
- Lyapunov functions in each mode
- Level sets are ellipses



Hybrid stability: Example 2

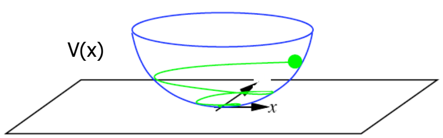
- These Lyapunov-like functions are acceptable candidates to show hybrid stability
- V(q,x) decreases while mode q is active
- V is lower when mode is re-visited

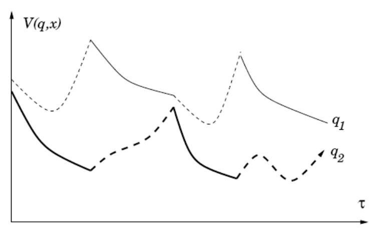




Multiple Lyapunov functions

- Often difficult to use in practice!
- To yield more practical theorems, focus on specific classes of
  - switching schemes (arbitrary, state-based, timed)
  - dwell times within each mode
  - continuous dynamics (linear, affine)
  - Lyapunov functions (common Lyapunov functions, piecewise quadratic Lyapunov functions, etc.)





Summary

- Hybrid equilibrium
  - Allows switching
  - Continuous state must be 0
- Hybrid stability
  - Switching allowed so long as the continuous state remains bounded
- Multiple Lyapunov functions
  - Most general form of stability
  - Difficult to use in practice
  - Narrow according to structure within the hybrid system