EECE 571M/491M, Spring 2008
Lecture 7

Global Quadratic Lyapunov Theory

Meeko Oishi, Ph.D.

Electrical and Computer Engineering

University of British Columbia, BC

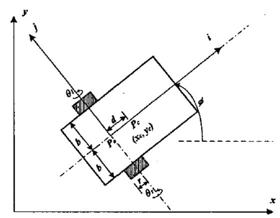
http://courses.ece.ubc.ca/491m

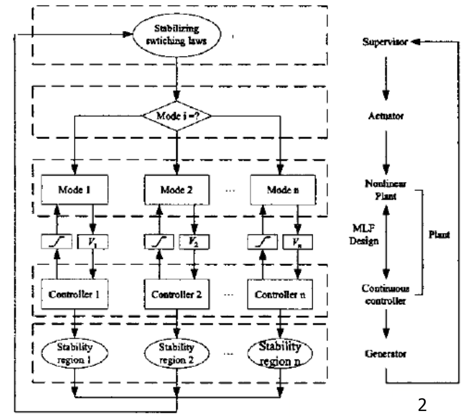
moishi@ece.ubc.ca

Tomlin LN 6, Johansson and Rantzer (1998)

MLF Example #2

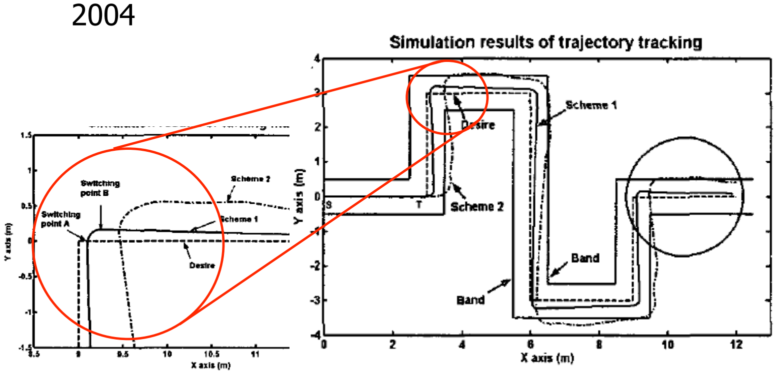
- Z. Zhang, N. Sarkar, X. Yun, "Supervisory control of a mobile robot for agile motion coordination," ICRA 2004.





MLF Example #2

- Z. Zhang, N. Sarkar, X. Yun, "Supervisory control of a mobile robot for agile motion coordination," ICRA 2004



MLF Example #2

- Z. Zhang, N. Sarkar, X. Yun, "Supervisory control of a mobile robot for agile motion coordination," ICRA 2004

$V(A1) \geq V(A2) \geq V(A3)$

$V(A4) \geq V(A5) \geq V(A6)$

but $V(A4) \geq V(A3)$

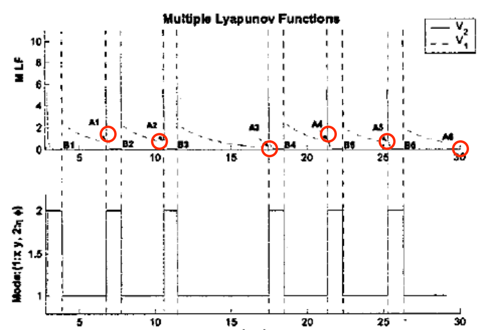


Fig. 5 Multiple Lyapunov Functions and switching sequence

Today's lecture

- Review -- ICRA example
  - Hybrid equilibrium
  - Hybrid stability
  - Multiple Lyapunov functions
- Global Linear Quadratic Lyapunov function
  - (Results from Johansson and Rantzer 1998)
- Converse GQLF theorem
  - (Results from Johansson and Rantzer 1998)

Globally Quad. Lyapunov Fcn

- Consider the case when
  - State-space is partitioned into disjoint regions
  - Reset map is the identity
  - Dynamics are linear and asymptotically stable in each mode
  - Only one transition is possible from each mode
- The same linear quadratic Lyapunov function is used in each mode (hence 'global')
  - $V_i(x) = x^T P x$ for all modes

  but $-Q_i = A_i^T P + P A_i$ may be different each mode
- Multiple Lyapunov Function theorem
  - is satisfied for $P > 0$, $Q_i > 0$.
  - can be simplified into the Globally Quadratic Lyapunov Function theorem.

Globally Quad. Lyapunov Fcn

Theorem: Globally Quadratic Lyapunov Function

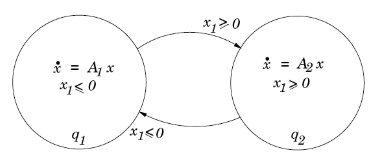
- Consider a hybrid automaton H = (Q, X, f, R, Init, Dom) with equilibrium x* = 0. Assume that
  - $f_i(x,t) = A_i x, \quad A \in \mathcal{R}^{n\times n}$
  - $|R(q_i,x)| = \begin{cases} 1 & \text{if } x \in \partial Dom \\ 0 & \text{otherwise} \end{cases}$
  - $R(q_i,x) \in Dom$
- If there exists a matrix $P = P^T > 0$ such that

$$A_i^T P + P A_i < 0$$

then x* = 0 is exponentially stable.

GQLF Example #1

- Consider the hybrid system



with state matrices $A_1 = \begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix}, A_2 = \begin{bmatrix} -2 & 1 \\ -1 & -2 \end{bmatrix}$

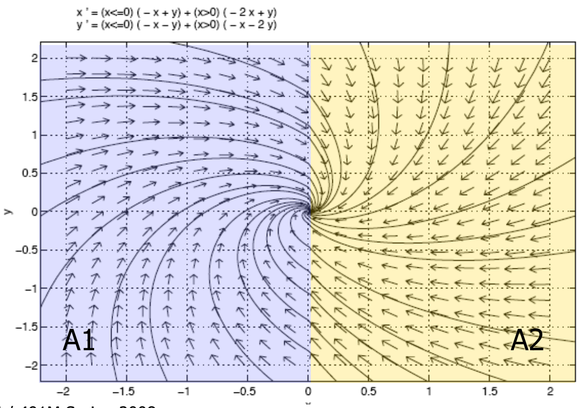
- Since $P = P^T = I$ results in

$$A_1^T + A_1 < 0 \text{ and } A_2^T + A_2 < 0$$

- $V = x^T x$ is a common Lyapunov function.

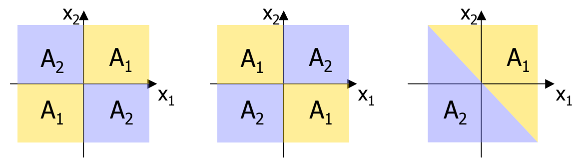
GQLF Example #1

- Phase plane plot $A_1 = \begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix}, A_2 = \begin{bmatrix} -2 & 1 \\ -1 & -2 \end{bmatrix}$



Globally Quad. Lyapunov Fcn

- Independent of the particular switching scheme



- Therefore will be stable under *arbitrary* switching schemes
- GQLF is provides **sufficient** condition for stability (i.e., if the GQLF exists, then the hybrid system is stable)

Globally Quad. Lyapunov Fcn

Theorem: Converse Globally Quadratic Lyapunov Function

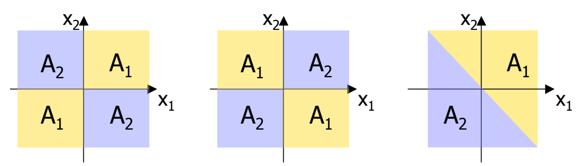
- Consider a hybrid automaton H = (Q, X, f, R, Init, Dom) with equilibrium x* = 0. Assume that
  - $f_i(x,t) = A_i x, \quad A \in \mathcal{R}^{n \times n}$
  - $|R(q_i, x)| = \begin{cases} 1 & \text{if } x \in \partial Dom \\ 0 & \text{otherwise} \end{cases}$
  - $R(q_i, x) \in Dom$
- If there exists matrices $\tilde{P}_i = \tilde{P}_i^T > 0$ such that

$$\sum_i A_i^T \tilde{P}_i + \tilde{P}_i A_i > 0$$

then **no** globally quadratic Lyapunov function exists.

Converse GQLF

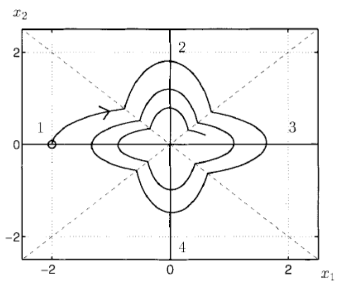
- Independent of the particular switching scheme

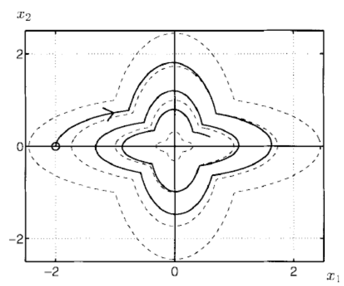


- Simple test to see whether a GQLF exists
- Can be solved as a LMI problem
- If GQLF does NOT exist, system may still be stable -- but other Lyapunov functions are required (e.g., multiple Lyapunov functions) to prove stability.

GQLF Example #2

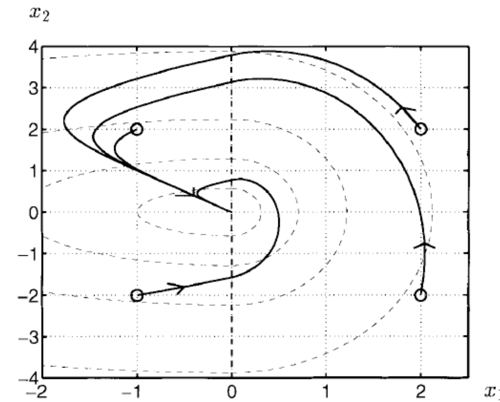
- Try A1 = [-0.1 1; -5 -0.1] =A3 and A2 = [-0.1 5; -1 -0.1] = A4 (stable, but no globally quadratic Lyapunov function)





GQLF Example #3

- Try A1 = [-5 -4; -1 -2] and A2 = [-2 -4; 20 -4] (stable, but no quadratic Lyapunov function)



Summary

- Global Quadratic Lyapunov functions
  - Easiest to start with
  - Irrespective of switching strategy
  - Solved by LMIs
- Converse global quadratic Lyapunov theorem
  - Test to determine whether GQLF exists
  - System may be stable even if test fails
  - Solved by LMIs
  - Irrespective of switching strategy

Relaxing the Global QLF

- Switched systems (continuity of the state)
- Only need to have positive definiteness within sectors (not R^n)
- Computationally done through “S-procedure”
- Still solve a set of LMIs

Piecewise Quad. Lyap. fun

- When no common Lyapunov function exists
- Different Lyapunov-like functions in each mode
- Disjoint partition of the state-space
  - Described by intersections of hyperplanes
  - Domain is set of convex polyhedra
- Linear dynamics in each mode

- Goal: Find P_i such that
  - $P_i > 0$ for x in X_i
  - $A^T P_i + P_i A < 0$ for x in X_i
  - and V(p,x) maintains continuity across modes

Piecewise Quad. Lyap. fun

Theorem 9 (Piecewise Quadratic Lyapunov Function) *$H = (S, \mathrm{Init}, f, \mathrm{Dom}, R)$ with equilibrium $x_e = 0$. Assume that for all i:*

- $f(q_i, x) = A_i x$, $A_i \in \mathbb{R}^{n \times n}$
- $\mathrm{Dom} = \cup_i \{q_i\} \times \{x \in \mathbb{R}^n : E_{i1} x \geq 0, \ldots, E_{in} x \geq 0\}$
- $\mathrm{Init} \subseteq \mathrm{Dom}$
- *for all $x \in \mathbb{R}^n$*
$$|R(q_i, x)| = \begin{cases} 1 & \text{if } (q_i, x) \in \partial\mathrm{Dom} \\ 0 & \text{otherwise} \end{cases} \tag{46}$$
such that
$$(q_k, x') \in R(q_i, x) \Rightarrow F_k x = F_i x, q_k \neq q_i, x' = x \tag{47}$$
where $F_k, F_i \in \mathbb{R}^{n \times n}$.

Furthermore, assume that for all $\chi \in \mathcal{E}_H^\infty$, $\tau_\infty(\chi) = \infty$. Then, if there exists $U_i = U_i^T$, $W_i = W_i^T$, and $M = M^T$ such that $P_i = F_i^T M F_i$ satisfies:

$$A_i^T P_i + P_i A_i + E_i^T U_i E_i < 0 \tag{48}$$
$$P_i - E_i^T W_i E_i > 0 \tag{49}$$

where U_i, W_i are non-negative, then $x_e = 0$ is asymptotically stable.

Piecewise Quad. Lyap. fun

$$A_1 = A_3 = \begin{bmatrix} -0.1 & 1 \\ -5 & -0.1 \end{bmatrix}, A_2 = A_4 = \begin{bmatrix} -0.1 & 5 \\ -1 & -0.1 \end{bmatrix}$$

$$E_1 = -E_3 = \begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix}, E_2 = -E_4 = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$$

$$F_i = \begin{bmatrix} E_i \\ I \end{bmatrix} \forall i \in \{1, 2, 3, 4\}$$

$$P_1 = P_3 = \begin{bmatrix} 5 & 0 \\ 0 & 1 \end{bmatrix}, P_2 = P_4 = \begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix}$$