EECE 571M/491M, Spring 2008
Lecture 9

Review and Examples: Linear Quadratic Lyapunov Theory

Meeko Oishi, Ph.D.

Electrical and Computer Engineering

University of British Columbia, BC

http://courses.ece.ubc.ca/491m

moishi@ece.ubc.ca

Johansson and Rantzer (1998), DeCarlo et al (2000)

Review

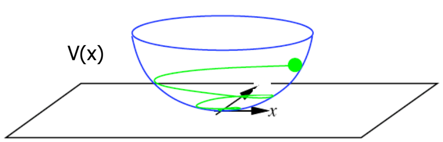
Three major theorems for hybrid systems:

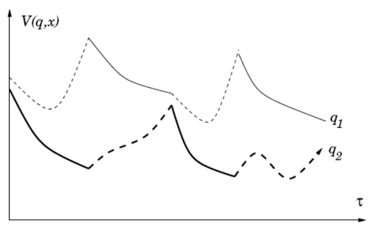
- Multiple Lyapunov Function
  - General nonlinear dynamics, non-identity reset map
- Globally Quadratic Lyapunov Function (a type of Common Lyapunov Function)
  - Linear dynamics, quadratic Lyapunov function
  - Identity reset map
  - SAME Lyapunov function must hold in ALL modes for ALL states (not just in domain)
- Piecewise Quadratic Lyapunov Function
  - Linear dynamics, quadratic Lyapunov function
  - Identity reset map
  - Potentially DIFFERENT Lyapunov functions in every mode, must hold only for states the DOMAIN of each particular mode.

Review: Multiple Lyap. Fcns.

Consider a Lyapunov-like function V(q,x):

- When the system is evolving in mode q, V(q,x) must decrease or maintain the same value
- Every time mode q is re-visited, the value V(q,x) must be lower than it was last time the system entered mode q.
- When the system switches into a new mode q', V may jump in value
- For inactive modes p, V(p,x) may increase
- Requires solving for V directly





Preview

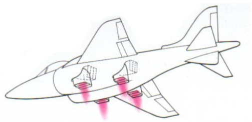
What's next:

- Return to Common Lyapunov Function
  - Classes of systems for which GQLFs are known to exist
  - Classes of systems for which GQLFs are known NOT to exist
- Switching control
  - How to use Lyapunov functions to synthesize a switching controller for stability
- Gain scheduling as hybrid control
- Chattering

Today's lecture

- Examples
  - Example #1: Switched stability of a VSTOL aircraft
    - M. Oishi and C. Tomlin, IEEE CDC 1999
  - Example #2: Backing up a truck-trailer
    - C. Altafini, A. Speranzon, K. Johansson, HSCC 2002
    - C. Altafini, A. Speranzon, B. Wahlberg, IEEE TRA 2001
  - Example #3: Biological cell dynamics
    - R. Ghosh, C. Tomlin, HSCC 2001

Example #1: VSTOL aircraft



"Four-poster" thrust-vectoring system

- Vectored thrust aircraft
- Three "modes" of operation
  - Conventional (CTOL)
  - Transition
  - Vertical (VTOL)
- Nonlinear system approximated by hybrid system
- Inputs
  - Thrust (T)
  - Pitch acceleration ($d^2\theta/dt^2$) (through the elevators)
  - Thrust angle acceleration ($d^2\delta/dt^2$) (through torque applied to the nozzles)
- State vector
  - Horizontal position x, horizontal speed dx/dt
  - Vertical position z, vertical speed dz/dt
  - Pitch θ, pitch rate $d\theta/dt$
  - Thrust angle δ, thrust angle rate $d\delta/dt$

Example #1: VSTOL aircraft

VTOL: Vertical Take-Off/Landing

- Thrust is 90^0 from horizontal
- Nonlinear dynamics

$$M\begin{bmatrix}\ddot{x}\\ \ddot{z}\end{bmatrix} = R(\theta)\begin{bmatrix}0\\ T-\epsilon u_2\end{bmatrix} - \begin{bmatrix}0\\ Mg\end{bmatrix}$$

- Closed-loop dynamics "linearized" by nonlinear control law which tracks desired trajectories in (x, z) (assuming $\varepsilon = 0$)
- Resulting error dynamics satisfy

$$\dot{e} = Ae, \text{ where} \quad A = \text{diag}\{A_1, A_2\} \quad A_1=\begin{bmatrix}0&1&0&0\\0&0&1&0\\0&0&0&1\\-\alpha_0^1&-\alpha_1^1&-\alpha_2^1&-\alpha_3^1\end{bmatrix} A_2=\begin{bmatrix}0&1&0&0\\0&0&1&0\\0&0&0&1\\-\alpha_0^2&-\alpha_1^2&-\alpha_2^2&-\alpha_3^2\end{bmatrix}$$

- With constants chosen by placing closed-loop poles at $\lambda_e = 1.3$,

$$(s+\lambda_e)^4 = s^4 + \alpha_3 s^3 + \alpha_2 s^2 + \alpha_1 s + \alpha_0$$

Example #1: VSTOL aircraft

CTOL: Conventional Take-Off/Landing

- Thrust is 0^0 from horizontal
- Nonlinear dynamics

$$M\begin{bmatrix}\ddot{x}\\ \ddot{z}\end{bmatrix} = R(\theta)\left(R^T(\alpha)\begin{bmatrix}-D\\ L\end{bmatrix} + \begin{bmatrix}T\\ -\epsilon u_2\end{bmatrix}\right) - \begin{bmatrix}0\\ Mg\end{bmatrix}$$

- Closed-loop dynamics "linearized" by nonlinear control law which tracks desired trajectories in (x, z) (assuming $\varepsilon = 0$)
- Resulting error dynamics satisfy

$$\dot{e} = Ae, \text{ where} \quad A = \text{diag}\{A_1, A_2\} \quad A_1=\begin{bmatrix}0&1&0&0\\0&0&1&0\\0&0&0&1\\-\alpha_0^1&-\alpha_1^1&-\alpha_2^1&-\alpha_3^1\end{bmatrix} A_2=\begin{bmatrix}0&1&0&0\\0&0&1&0\\0&0&0&1\\-\alpha_0^2&-\alpha_1^2&-\alpha_2^2&-\alpha_3^2\end{bmatrix}$$

- With constants chosen to place closed-loop poles at $\lambda_e = 1.3$,

$$(s+\lambda_e)^4 = s^4 + \alpha_3 s^3 + \alpha_2 s^2 + \alpha_1 s + \alpha_0$$

Example #1: VSTOL aircraft

TRANSITION: Short Take-Off/Vertical Landing

- Thrust is 0^0-90^0 from horizontal
- Nonlinear dynamics

$$M\begin{bmatrix}\ddot{x}\\ \ddot{z}\end{bmatrix} = R(\theta)\left(R^T(\alpha)\begin{bmatrix}-D\\ L\end{bmatrix}+\begin{bmatrix}T\cos\delta\\ T\sin\delta-\epsilon u_2\end{bmatrix}\right)-\begin{bmatrix}0\\ Mg\end{bmatrix}$$

- Closed-loop dynamics "linearized" by nonlinear control law which tracks desired trajectories in (x, z, δ) (assuming ε = 0)
- Resulting error dynamics satisfy

$$\dot{e} = Ae, \text{ where} \qquad A = \operatorname{diag}(A_1, A_2, A_3)$$

$$A_1=\begin{bmatrix}0&1&0&0\\0&0&1&0\\0&0&0&1\\-\alpha_0^1&-\alpha_1^1&-\alpha_2^1&-\alpha_3^1\end{bmatrix} \quad A_2=\begin{bmatrix}0&1&0&0\\0&0&1&0\\0&0&0&1\\-\alpha_0^2&-\alpha_1^2&-\alpha_2^2&-\alpha_3^2\end{bmatrix}$$

- With constants chosen to place closed-loop poles at $\lambda_e = 1.3,\ \lambda_d = 1.3$

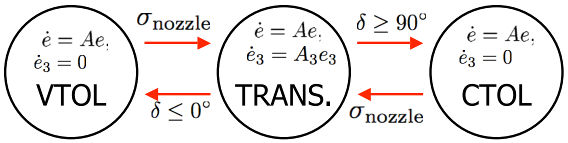
$$A_3 = \begin{bmatrix}0&1\\-\alpha_0^3&-\alpha_1^3\end{bmatrix}$$

$$(s+\lambda_e)^4 = s^4+\alpha_3 s^3+\alpha_2 s^2+\alpha_1 s+\alpha_0$$

$$(s+\lambda_d)^2 = s^2+\alpha_1^3 s+\alpha_0^3$$

Example #1: VSTOL aircraft

- Switched nonlinear system
- "Placeholder" continuous variables in modes with fewer states



- Now find a linear quadratic Lyapunov function to guarantee stability of the switched system
- Easiest to start with a Common Lyapunov function

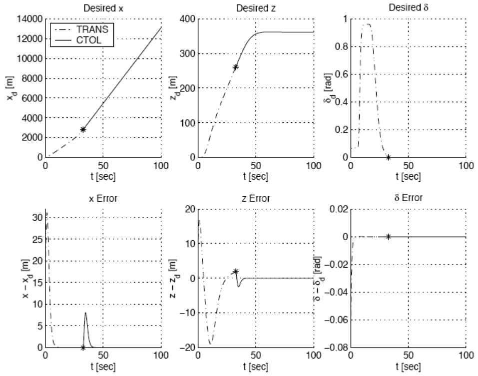
$$v(e,t) = e^T P e \qquad P > 0$$

$$\overline{A}^T P + P\overline{A} = -I \qquad \overline{A} = \operatorname{diag}\{A, A_3\}$$

- Which we know exists since $\overline{A}$ is stable.

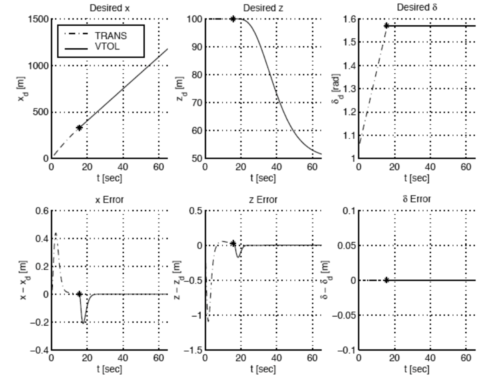
Example #1: VSTOL aircraft

- A Common Lyapunov function guarantees stability for **any** switching strategy
- Short Take-off Maneuver: (TRANSITION -> CTOL)



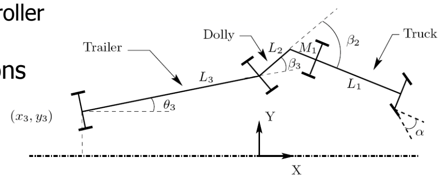
Example #1: VSTOL aircraft

- A Common Lyapunov function guarantees stability for **any** switching strategy
- Vertical Landing Maneuver: (TRANSITION -> VTOL)



Example #2: 3-axle trailer

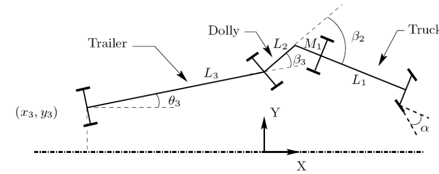
- Experimental truck and trailer
- 1:16 scale model of commercial vehicles
- Unstable nonlinear dynamics
- Nonholonomic system (non-integrable constraints)
- Goal: back up along a specified path
- Control not possible with a single, nonlinear controller
- Hierarchical, hybrid system:
  - Straight-line tracking controller (forwards and backwards)
  - Arc-tracking controller
- Linearize dynamics around these motions



Example #2: 3-axle trailer

- Straight line tracking

$$\dot{p} = vAp + B\alpha$$
$$p = [y_3, \theta_3, \beta_3, \beta_2]^T$$



$$A = \left.\frac{\partial \mathcal{A}(\mathbf{p})}{\partial \mathbf{p}}\right|_{(0)} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1/L_3 & 0 \\ 0 & 0 & -1/L_3 & 1/L_2 \\ 0 & 0 & 0 & -1/L_2 \end{bmatrix}, \quad B = \left.\frac{\partial \mathcal{B}(\mathbf{p},\, \alpha)}{\partial \mathbf{p}}\right|_{(0,\,0)} = \begin{bmatrix} 0 \\ 0 \\ -M_1/(L_1 L_2) \\ (L_2 + M_1)/(L_1 L_2) \end{bmatrix}$$

- This mode is open-loop stable in forward motion (v > 0)

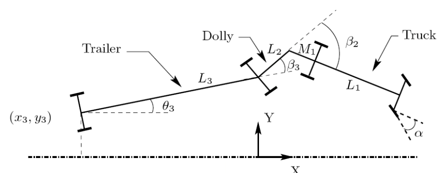
$$\det(sI - vA) = s^2 \left(s + \frac{v}{L_2}\right)\left(s + \frac{v}{L_3}\right)$$

- But open-loop unstable in backwards motion (v < 0)

Example #2: 3-axle trailer

- Arc-tracking

$$\dot{p} = vAp + B\alpha$$
$$p = [\beta_3, \beta_2]^T$$



- With a non-zero equilibrium point,

$$\beta_{2e} = \arctan\left(\frac{M_1}{r_1}\right) + \arctan\left(\frac{L_2}{r_2}\right), \qquad \beta_{3e} = \arctan\left(\frac{r_3}{L_3}\right)$$

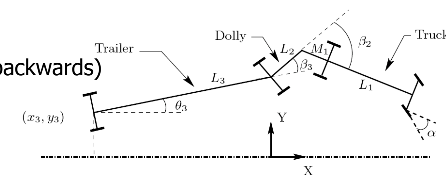
- The linearized dynamics are $\dot{\bar{\mathbf{p}}} = v\big(\bar{A}(\bar{\mathbf{p}} - \bar{\mathbf{p}}_e) + \bar{B}(\alpha - \alpha_e)\big)$

$$\bar{A} = \begin{bmatrix} \dfrac{\cos\beta_{2e}\cos\beta_{3e}}{L_3} & \dfrac{\cos\beta_{2e}}{L_2} + \dfrac{\sin\beta_{2e}\sin\beta_{3e}}{L_3} + \dfrac{M_1}{L_1}\left(\dfrac{\sin\beta_{2e}}{L_2} - \dfrac{\cos\beta_{2e}\sin\beta_{3e}}{L_3}\right)\tan\alpha_e \\ 0 & -\dfrac{\cos\beta_{2e}}{L_2}\left(1 + \dfrac{M_1}{L_1}\tan\beta_{2e}\tan\alpha_e\right) \end{bmatrix}$$

$$\bar{B} = \begin{bmatrix} -\dfrac{M_1}{L_1}\left(\dfrac{\cos\beta_{2e}}{L_2} + \dfrac{\sin\beta_{2e}\sin\beta_{3e}}{L_3}\right)(1 + \tan^2\alpha_e) \\ \dfrac{1}{L_1}\left(1 + \dfrac{M_1}{L_2}\cos\beta_{2e}\right)(1 + \tan^2\alpha_e) \end{bmatrix}$$

Example #2: 3-axle trailer

- Jack-knife configuration
(Trailer cannot be pushed backwards)

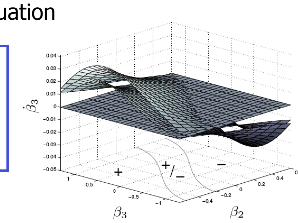


- Constraints

$$|\beta_2| \le \beta_{2s} = 0.6 \text{ rad},$$
$$|\beta_3| \le \beta_{3s} = 1.3 \text{ rad}$$

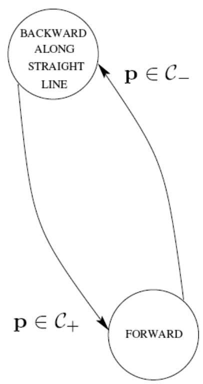
- Encode constraints into the domain, to ensure system is not allowed to evolve into jack-knife situation

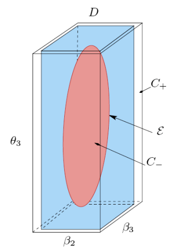
- Goal: Design a switched controller to move backwards along a desired path, e.g, stabilize tracking error



Example #2: 3-axle trailer

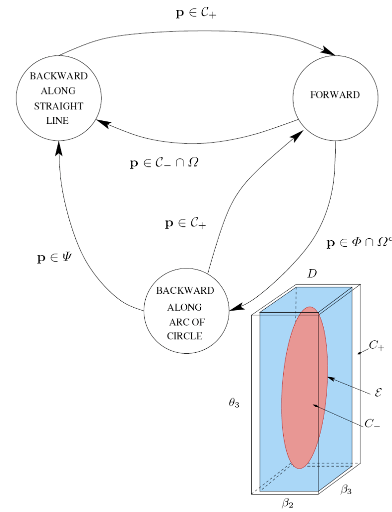
- Scenario 1: Two modes
- Switching surfaces C_- and C_+
- Results in unacceptably slow convergence of the state θ_3 (trailer alignment)
- Solution: Try adding a third mode, to improve alignment





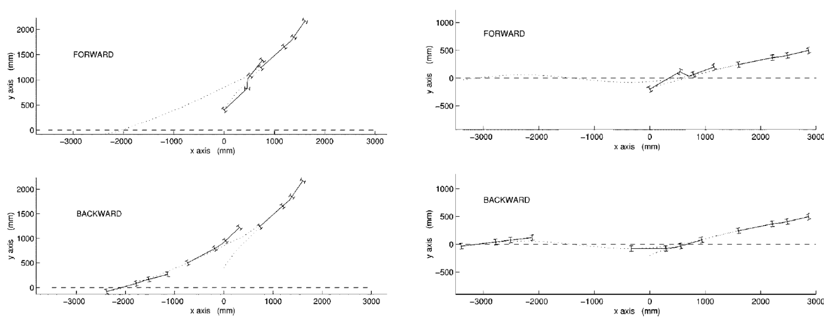
Example #2: 3-axle trailer

- Scenario 2: Three modes
- Switching surfaces C_-, C_+, Ω, Φ, Ψ
- Choosing these surfaces is part of the control design problem (regions of attraction)
- Surfaces must be chosen to satisfy conditions for hybrid stability.



Example #2: 3-axle trailer

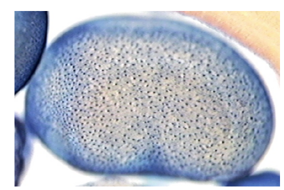
- Initial conditions near the jack-knife scenario



Example #3: Delta-Notch

- Cell differentiation is process by which a group of cells initially of homogenous composition becomes heterogeneous.
- Cell type is determined by genes, which determine
  - Amount of protein produced
  - Type of protein produced

  in a given cell



- Proteins affect gene activity by
  - Turning "on" gene expression
  - Turning "off" gene expression
- Feedback is an inherent part of the biological mechanism of cell differentiation

Example #3: Delta-Notch

- "Delta" and "Notch" are two key proteins involved in cell differentiation
  - Inter-cellular signaling (within a single cell)
  - Intra-cellular signaling (across neighboring cell boundaries)
- Experimentally observed "rules" of Delta-Notch interaction
  - High Delta levels triggers production of Notch in neighboring cells
  - Low Notch levels triggers production of Delta in the same cell
  - High Delta levels produce differentiated cells
  - Low Delta levels produce undifferentiated cells
  - Delta and Notch decay exponentially

Example #3: Delta-Notch

- Piecewise affine model
- Each cell can be in one of four modes
  - Delta decaying, Notch decaying
  - Delta produced, Notch decaying
  - Delta decaying, Notch produced
  - Delta produced, Notch produced
- The continuous state is the concentrations of Delta, Notch (v_D,v_N)
- Switching thresholds (h_D, h_N) trigger Delta, Notch production
- Inputs (u_D, u_N) states in neighboring cells
- Constants
  - Decay rates (λ_D, λ_N)
  - Production rates (R_D, R_N)

Example #3: Delta-Notch

- Hybrid automaton

$$
\begin{aligned}
H_1 &= (Q_1, X_1, \Sigma_1, V_1, Init_1, f_1, Inv_1, R_1)\\
Q_1 &= \{q_1, q_2, q_3, q_4\}\\
X_1 &= (v_D, v_N)^T \in \mathbb{R}^2\\
\Sigma_1 &= \left\{u_D, u_N : u_D = -v_N, u_N = \sum_{i=1}^{6} v_D^i\right\}\\
V_1 &= \emptyset\\
Init_1 &= Q_1 \times \{X_1 \in \mathbb{R}^2 : v_D, v_N > 0\}\\
f_1(q,x) &= \begin{cases} [-\lambda_D v_D; -\lambda_N v_N]^T & \text{if } q = q_1\\ [R_D - \lambda_D v_D; -\lambda_N v_N]^T & \text{if } q = q_2\\ [-\lambda_D v_D; R_N - \lambda_N v_N]^T & \text{if } q = q_3\\ [R_D - \lambda_D v_D; R_N - \lambda_N v_N]^T & \text{if } q = q_4 \end{cases}\\
Inv_1 &= \{q_1, \{u_D < h_D, u_N < h_N\}\} \cup \{q_2, \{u_D \geq h_D, u_N < h_N\}\}\\
&\quad \cup \{q_3, \{u_D < h_D, u_N \geq h_N\}\} \cup \{q_4, \{u_D \geq h_D, u_N \geq h_N\}\}
\end{aligned}
$$

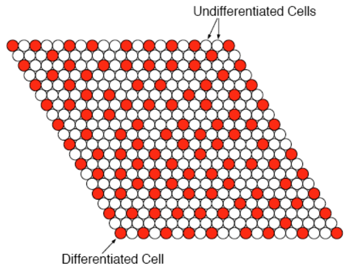
Example #3: Delta-Notch

- Hybrid automaton

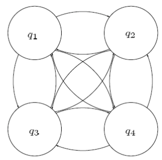
$$
R_1 \; : \; \begin{bmatrix}
R_1\left(q_1, \{u_D \geq h_D \wedge u_N < h_N\}\right) \in q_2 \times \mathbb{R}^2\\
R_1\left(q_1, \{u_D < h_D \wedge u_N \geq h_N\}\right) \in q_3 \times \mathbb{R}^2\\
R_1\left(q_1, \{u_D \geq h_D \wedge u_N \geq h_N\}\right) \in q_4 \times \mathbb{R}^2\\
R_1\left(q_2, \{u_D < h_D \wedge u_N < h_N\}\right) \in q_1 \times \mathbb{R}^2\\
R_1\left(q_2, \{u_D < h_D \wedge u_N \geq h_N\}\right) \in q_3 \times \mathbb{R}^2\\
R_1\left(q_2, \{u_D \geq h_D \wedge u_N \geq h_N\}\right) \in q_4 \times \mathbb{R}^2\\
R_1\left(q_3, \{u_D < h_D \wedge u_N < h_N\}\right) \in q_1 \times \mathbb{R}^2\\
R_1\left(q_3, \{u_D \geq h_D \wedge u_N < h_N\}\right) \in q_2 \times \mathbb{R}^2\\
R_1\left(q_3, \{u_D \geq h_D \wedge u_N \geq h_N\}\right) \in q_4 \times \mathbb{R}^2\\
R_1\left(q_4, \{u_D < h_D \wedge u_N < h_N\}\right) \in q_1 \times \mathbb{R}^2\\
R_1\left(q_4, \{u_D \geq h_D \wedge u_N < h_N\}\right) \in q_2 \times \mathbb{R}^2\\
R_1\left(q_4, \{u_D < h_D \wedge u_N \geq h_N\}\right) \in q_3 \times \mathbb{R}^2
\end{bmatrix}
$$

Example #3: Delta-Notch

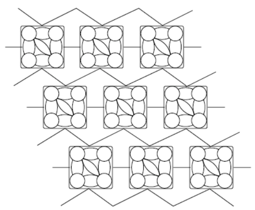
- Simulation based on experimentally determined constants
- Consistent with known data



Single cell



3x3 cells



Example #3: Delta-Notch

- Problem:
  - What are the switching thresholds?
  - What are the equilibria possible (besides experimentally known)?
  - What initial conditions reach those equilibria?

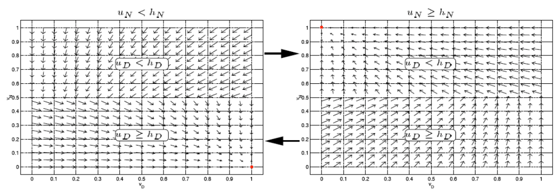
- Special case: One cell

Mode	Equilibrium	Existence condition	Label
q_1	$v_D^* = 0, v_N^* = 0$	$0 < h_D \wedge u_N < h_N$	$\boxed{\text{dead cell}}$
q_2	$v_D^* = \frac{R_D}{\lambda_D}, v_N^* = 0$	$0 \geq h_D \wedge u_N < h_N$	differentiated cell
q_3	$v_D^* = 0, v_N^* = \frac{R_N}{\lambda_N}$	$-\frac{R_N}{\lambda_N} < h_D \wedge u_N \geq h_N$	undifferentiated cell
q_4	$v_D^* = \frac{R_D}{\lambda_D}, v_N^* = \frac{R_N}{\lambda_N}$	$-\frac{R_N}{\lambda_N} \geq h_D \wedge u_N \geq h_N$	$\boxed{\text{“confused” cell}}$

- Provides constraints on thresholds for existence of biologically meaningful equilibrium points

Example #3: Delta-Notch

- Special case: One cell



- Two biologically meaningful equilibria:
  - Differentiated
  - Undifferentiated

  depending on whether $u_N < h_N$ or $u_N > h_N$

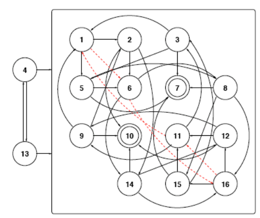
- But this does not capture full behavior -- need more cells interacting.

Example #3: Delta-Notch

- Special case: Two cells
- Only 6 out of 16 equilibria are biologically feasible
- Only 2 of the remaining 6 are not mutually exclusive
- Choosing

$$h_D, h_N : -\frac{R_N}{\lambda_N} < h_D \leq 0 \wedge 0 < h_N \leq \frac{R_D}{\lambda_D}$$

enables the two possible equilibria in the 2-cell automaton



- Larger NxN cell array requires more robust analysis (model checking)

Summary

- Review of the main Lyapunov theorems so far
- Examples of hybrid systems
  - Stabilizing a VSTOL aircraft
  - Backing up a tractor-trailer
  - Biological circuits
- Next few classes:
  - Discrete-time piecewise affine Lyapunov theorems
  - Polynomial dynamics and Sum-of-squares Lyapunov functions
  - Special cases of common Lyapunov functions