EECE 571M/491M, Spring 2008

Lecture 10

Common Lyapunov Functions

Meeko Oishi, Ph.D.

Electrical and Computer Engineering

University of British Columbia, BC

http://courses.ece.ubc.ca/491m

moishi@ece.ubc.ca

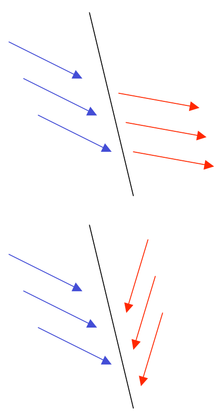
Liberzon and Morse, 1999.

Today's lecture

- Chattering and Sliding modes
- Common Lyapunov functions for specific classes of switched linear systems
  - Commuting system matrices
  - Upper-triangular system matrices
  - Two-dimensional system matrices
- Control design for stability
  - Introduction

Sliding Modes

- Implicit assumption that piecewise-linear dynamics do not chatter
- However, attractive 'sliding modes' are possible
- Proving stability of a sliding mode is more difficult, but still possible
- Can also be formulated as LMI constraint
- Detection of sliding modes

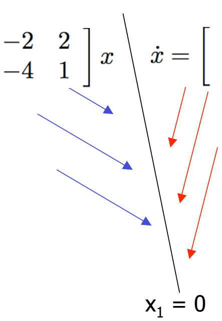


Sliding Modes

- Requires extension of solution in the sense of Filippov
- (Solution lies within convex hull of dynamics)
- Define a 'sliding surface' of width ε --> 0
- Show decreasing Lyapunov function along sliding surface

Example #3:

$\dot{x} = \begin{bmatrix} -2 & 2 \\ -4 & 1 \end{bmatrix} x \qquad \dot{x} = \begin{bmatrix} -2 & -2 \\ 4 & 1 \end{bmatrix} x$

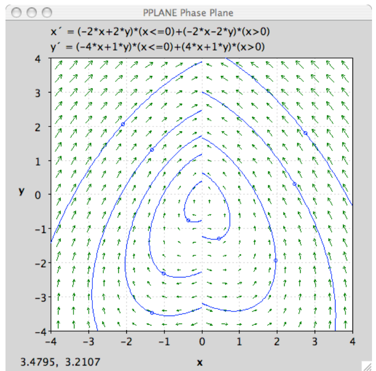


Sliding Modes

- Trajectories appear to 'stop' in pplane
- Integration not possible.

$$\dot{x} = \begin{bmatrix} -2 & 2 \\ -4 & 1 \end{bmatrix} x \quad x_1 <= 0$$

$$\dot{x} = \begin{bmatrix} -2 & -2 \\ 4 & 1 \end{bmatrix} x \quad x_1 >= 0$$



Sliding Modes

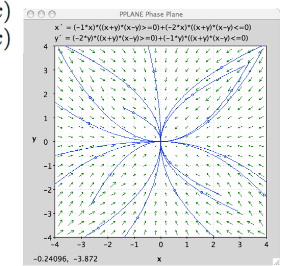
- Midterm Problem #3

$$\dot{x} = A_i x,\ A_1 = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix},\ A_2 = \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix},$$

$$\mathrm{Dom} = (q_1, \{x \mid (x_1 + x_2)(x_1 - x_2) \geq 0\} \cup (q_2, \{x \mid (x_1 + x_2)(x_1 - x_2) \leq 0\}),$$

$$\begin{aligned} R(q_1, \{x \mid (x_1 + x_2)(x_1 - x_2) \leq 0\} &= (q_2, x) \\ R(q_2, \{x \mid (x_1 + x_2)(x_1 - x_2) \geq 0\} &= (q_1, x) \end{aligned}$$

- Hybrid system is stable
- No chattering occurs



Sliding Modes

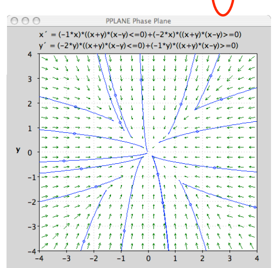
- REVISED Midterm Problem #3

$$\dot{x} = A_i x,\ A_1 = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix},\ A_2 = \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix},$$

$$\mathrm{Dom} = (q_1, \{x \mid (x_1 + x_2)(x_1 - x_2) \leq 0\} \cup (q_2, \{x \mid (x_1 + x_2)(x_1 - x_2) \geq 0\}),$$

$$\begin{aligned} R(q_1, \{x \mid (x_1 + x_2)(x_1 - x_2) \geq 0\} &= (q_2, x) \\ R(q_2, \{x \mid (x_1 + x_2)(x_1 - x_2) \leq 0\} &= (q_1, x) \end{aligned}$$

- Note change in domain and transition function
- Hybrid system is stable, assuming sol'n in the sense of Filipov
- Chattering DOES occur
- Trajectories get 'stuck' at $x_1=x_2$ and $x_1=-x_2$



Common Lyapunov Functions

- So far
  - Quadratic Common Lyapunov function for switched linear systems
- Goal: Identify specific classes of systems for which common Lyapunov functions exist
- Arbitrarily switched linear systems with:
  - Commuting system matrices
  - Upper-triangular system matrices
  - Two-dimensional system matrices

Commuting system matrices

- Consider a two-mode system for which
  $$A_1 A_2 = A_2 A_1$$

$$e^{A_1}e^{A_2} = e^{A_1+A_2}$$
$$e^{A_1\tau}e^{A_2\rho} = e^{A_2\rho}e^{A_1\tau}$$

- Label time intervals in mode 1, mode 2 by ρ_i, τ_i, respectively.
- The state trajectory after 2*n* mode transitions at some time *t* is:

$$\begin{aligned} x(t) &= e^{A_2\tau_n}e^{A_1\rho_n}\cdots e^{A_2\tau_2}e^{A_1\rho_2}e^{A_2\tau_1}e^{A_1\rho_1}\cdot x(0) \\ &= e^{A_2\tau_n}\cdots e^{A_2\tau_1}\cdot e^{A_1\rho_n}\cdots e^{A_1\rho_1}\cdot x(0) \\ &= e^{A_2(\tau_n+\cdots+\tau_1)}\cdot e^{A_1(\rho_n+\cdots+\rho_1)}\cdot x(0) \end{aligned}$$

- And as $t \to \infty$, $\sum_i \tau_i \to \infty$ or $\sum_i \rho_i \to \infty$
- Therefore $x(t) \to 0$ as $t \to \infty$

Commuting system matrices

- **Theorem:**
  If a switched linear system
  $$\dot{x} = A_i x, \quad i \in \mathcal{I}$$
  has system matrices A_i that commute and are Hurwitz, then the arbitrarily switched linear system is stable.
- A quadratic Common Lyapunov Function is

$$V(x) = x^T P_m x, \text{ where } m = |\mathcal{I}|, \text{ and}$$
$$\begin{aligned} -I &= A_1^T P_1 + P_1 A_1 \\ -P_{i-1} &= A_i^T P_i + P_i A_i \\ &\vdots \\ -P_{m-1} &= A_m^T P_m + P_m A_m \end{aligned}$$

Commuting system matrices

- Example:

$$\dot{x} = A_i x,\ A_1 = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix},\ A_2 = \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix},$$

- Check: $A_1 A_2 = A_2 A_1$

$$\begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}\begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$
$$\begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

- And eig(A_1) = {-1,-2}, eig(A_2) = {-2,-1} (both are Hurwitz)
- **Diagonal matrices commute.**
- What other classes of matrices commute?

Upper-tri. system matrices

- **Theorem:**
  If a switched linear system
  $$\dot{x} = A_i x, \quad i \in \mathcal{I}$$
  has system matrices A_i that are upper-triangular and are Hurwitz, then the arbitrarily switched linear system is exponentially stable.
- (Proof:
  Solvable Lie Algebras all have transformations to upper-triangular form.
  Switched systems with solvable Lie Algebras and Hurwitz system matrices are exponentially stable.)

Upper-tri. system matrices

- Example:

$$\dot{x} = A_i x, \quad A_1 = \begin{bmatrix} -1 & 2 \\ 0 & -3 \end{bmatrix}, A_2 = \begin{bmatrix} -2 & 3 \\ 0 & -4 \end{bmatrix}$$

- $\mathrm{Eig}(A_1) = \{-1, -3\}$
- $\mathrm{Eig}(A_2) = \{-2, -4\}$

Two-dim. two-mode system

- **Theorem:**
  A two-dimensional, two-mode arbitrarily switched linear system
  $$\dot{x} = A_i x, \quad i \in \{1, 2\}$$
  is asymptotically stable **if and only if** all pair-wise convex combinations of
  $$A_1, A_2, A_1^{-1}, A_2^{-1}$$
  are Hurwitz.

- Related Theorem:
  If a switched system is asymptotically stable, then all convex combinations of all subsystem pairs are globally asymptotically stable.

- Recall that a convex combination of two matrices, *A, B* is
  $$\alpha A + (1-\alpha) B, \quad \alpha \in [0, 1]$$

Common similarity transf.

- **Theorem:**
  If for a switched linear system
  $$\dot{x} = A_i x, \quad i \in \mathcal{I}$$
  with system matrices A_i that are Hurwitz, and
  there exists a common nonsingular matrix T such that the matrices
  $$\tilde{A}_i = T^{-1} A_i T$$
  are upper-triangular, then the arbitrarily switched linear system is exponentially stable.

- A quadratic Common Lyapunov Function is
  $$V(z) = z^T P z, \quad P \triangleq T^{-T} P T^{-1}$$
  for which the transformed coordinates in each mode are
  $$z = Tx$$

Common similarity transf.

- Note that if A and B are diagonalizable, they share the same eigenvector matrix T **if and only if** AB=BA.

- Note that a diagonal matrix $\Lambda_i = T^{-1} A_i T$ is also upper-triangular.

Introduction to Hybrid Control

- **Problem 1:**
- Given a switched linear system with Hurwitz matrices, what are the classes of switching signals for which the switched system is stable?
  (Systems for which a common Lyapunov function does not exist)
  - Dwell times
  - Slow switching
- **Problem 2:**
- Given a switched linear system with NO Hurwitz matrices, construct a switching signal that stabilizes the switched system.
  (Systems which contain a stable subsystem can be solved trivially)
  - Existence of a stabilizing switching signal
  - Estimation-based supervisors

Introduction to Hybrid Control

- **Problem 3:**
- Given a non-autonomous hybrid system, how should the continuous and discrete inputs be chosen to ensure a stable and/or optimal closed-loop hybrid system?
  - Stabilizing model predictive control (MPC)
  - Optimal control of hybrid systems with terminal constraints

Two-mode switched linear sys.

- **Theorem:**
  There exists a stabilizing switching scheme such that the linear system
  $$\dot{x} = A_i x, \quad i \in \{1, 2\}$$
  (with unstable A_i) is asymptotically stable **if and only if** there exists α in (0,1) such that
  $$A_{eq} = \alpha A_1 + (1-\alpha) A_2$$
  is Hurwitz.
- The piecewise Lyapunov function which proves stability:
  $$V(q,x) = x^T P_{eq} x, \quad A_{eq}^T P_{eq} + P_{eq} A_{eq} = -Q, \ \text{for some } Q > 0$$
- The switching scheme is enforced through the state-space partition given by:
  $$\mathrm{Dom} = \cup_i \left(q_i, \{x \mid x^T (A_i^T P_{eq} + P_{eq} A_i) x < 0\}\right),$$

Switched linear systems

- **Theorem:**
  If for the switched linear system
  $$\dot{x} = A_i x, \quad i \in \{1, 2, \cdots, m\}$$
  there exists a stable convex combination of all state matrices, e.g.
  $$A_{eq} = \sum_{i=1}^{m} \alpha_i A_i, \quad \alpha_i > 0, \ \sum_i \alpha_i = 1$$
  then there exists a stabilizing switching scheme
  $$\sigma(x) = \arg\min_{i \in I} x^T (A_i^T P_{eq} + P_{eq} A_i) x$$
  with piecewise quadratic Lyapunov function
  $$V(q,x) = x^T P_{eq} x, \quad A_{eq}^T P_{eq} + P_{eq} A_{eq} = -Q, \ \text{for some } Q > 0$$
- Note that for *m* > 2 this provides **sufficient** (not necessary and sufficient) conditions.

Summary

- Common Lyapunov Theorems for switched linear systems
  - Commuting system matrices
  - Upper-triangular system matrices
  - Two-dimensional system matrices
  - Common similarity transformations

- Control synthesis for stability
  - Introduction