On $\mathcal{H}_\infty$ Control for Dead-Time Systems

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***Abstract*—A mixed sensitivity $\mathcal{H}_\infty$ problem is solved for dead-time systems. It is shown that for a given bound on the $\mathcal{H}_\infty$-norm causal stabilizing controllers exist that achieve this bound if and only if a related finite-dimensional Riccati equation has a solution with a certain nonsingularity property. In the case of zero time delay, the Riccati equation is a standard Riccati equation and the nonsingularity condition is that the solution be nonnegative definite. For nonzero time delay, the nonsingularity condition is more involved but still allows us to obtain controllers. All suboptimal controllers are parameterized, and the central controller is shown to be a feedback interconnection of a finite-dimensional system and a finite memory system, both of which can be implemented. Some $\mathcal{H}_\infty$ problems are rewritten as pure rational $\mathcal{H}_\infty$ problems using a Smith predictor parameterization of the controller.**

***Index Terms*—Dead-time systems, delay systems, $\mathcal{H}_\infty$ control, infinite-dimensional systems, Riccati equations, Smith predictors, spectral factorization.**

I. Introduction

DEAD-TIME systems are systems in which the action of control inputs takes a certain time before it affects the measured outputs. The typical dead-time system is

$$e^{-s\tau}P_r(s), \tag{1}$$

where P_r is some rational function and τ is a positive delay. Models like these appear frequently in applications for several reasons. One reason is the abundance of delay systems in real life, such as systems with transport delay. Another reason is that they often serve as a simple yet adequate model for otherwise complicated high-order or infinite-dimensional systems. From a mathematical system theory point of view, dead-time systems are infinite-dimensional, meaning that their state is an infinite-dimensional vector. Because they have a simple transfer function, however, they lend themselves well for analysis and controller design.

Controllers only based on the rational part of the dead-time system (1) generally do not work if the dead-time τ is large. Therefore, a need exists for controller design specific for this class. The first to design a controller that took into account the dead-time was Smith. In his paper [27] from 1957, he constructed a controller that achieved a complementary sensitivity function equal to a desired one times $e^{-s\tau}$. The desired complementary sensitivity was designed on the bases of P_r only. Hence, he transformed the controller design problem for the dead-time system into a rational controller design problem. Since Smith's seminal work, many generalizations and modification have been put forward; see, for instance, [35] and the reference therein. The (modified) Smith predictors were mainly designed to achieve good constant reference signal tracking and good constant disturbance signal rejection.

There is a long-lasting discussion regarding the robustness of the Smith predictor. Without giving a definite answer, we indicate in Section IV that the Smith predictor has the same robustness with respect to additive perturbations as the rational system. If the system is unstable, then the rational system has to be replaced by a modified (rational) plant that depends on the dead-time. The robustness margin is thus influenced by the dead-time.

In the 1980's, the $\mathcal{H}_\infty$ control problem became popular. In a few words, the $\mathcal{H}_\infty$ control problem is to find a controller that stabilizes a system and minimizes the $\mathcal{H}_\infty$-norm of an associated transfer function. A plethora of approaches exist to finite-dimensional $\mathcal{H}_\infty$ control theory. Among these, we categorize three that have had some bearing on the infinite-dimensional case. These approaches are as follows.

Operator-Theoretic Methods: See, e.g., Ball and Helton [1]. For the dead-time systems, see Foias *et al.* [10], Foias *et al.* [11], Özbay *et al.* [22], Toker and Özbay [32], and Zhou and Khargonekar [38]. The book [10] treats a general class of infinite-dimensional $\mathcal{H}_\infty$ control problems from an operator theoretic perspective. Via a series of conversions, a mixed sensitivity $\mathcal{H}_\infty$ problem is brought back to a two-block problem, which is then solved. Toker and Özbay [32] showed that the overall algorithm can be simplified significantly. The approach determines the optimal controller for single-input–single-output (SISO) dead-time systems.

Also of interest is the paper by Dym *et al.* [7]. They provide explicit controller formulas for a gap metric problem of dead-time systems, known to be equivalent to a special case of the mixed sensitivity $\mathcal{H}_\infty$ control problem. The gap metric problem is special in that it can be expressed as an optimal Hankel norm approximation problem. Building on work by Partington and Glover [24], they are able to construct the optimal controller using state-space realizations as a computational tool.

One way to calculate the optimal $\mathcal{H}_\infty$-norm is to use the essential spectrum of a certain Hankel plus Toeplitz operator. The result is this: if an eigenvalue of that operator exists whose absolute value is larger than the essential spectral radius, then the norm is the largest eigenvalue. Otherwise, the norm is the essential spectral radius. Because the essential spectral radius is not changed under compact perturbations, the essential spectral radius can be found by considering a simpler operator; see, e.g., [9], [16], [37], and [38].

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State-Space Methods: See, e.g., the DGKF paper [6] (finite-dimensional) and van Keulen [33] (infinite-dimensional). For the dead-time systems, see Kojima and Ishjima [14], Nagpal and Ravi [21], and Tadmor [28]–[31]. In [21] and [29], the infinite-dimensional problem is reduced to a finite-dimensional problem.

J-Spectral Factorization Methods: This approach has some overlap with operator theoretic methods. See, e.g., Green *et al.* [12], Kwakernaak [15], and Meinsma [17] (finite-dimensional), Curtain and Green [4] (infinite-dimensional). In Curtain and Green [4], several infinite-dimensional $\mathcal{H}_\infty$ problems were solved, but only on a very general level. For a specific transfer function, it is not clear how to solve the necessary equations and obtain an explicit controller.

We solve the mixed sensitivity $\mathcal{H}_\infty$ control problem for dead-time systems. Our approach follows the J-spectral factorization approach. Like with the Smith predictor, we apply a transformation that reduces the problem to a rational problem. Our (central) controller resembles the Smith predictor in that it is a rational system in feedback with a system whose impulse response has compact support. This very fact makes simulation and implementation of such controllers possible.

Our approach gives (sub)optimal controllers for multiple-input–multiple-output (MIMO) systems. The calculations needed to construct these controllers are all matrix calculations, involving a finite-dimensional Riccati equation, and the method is easy to implement in, e.g., MATLAB. In [40], the results of this paper were used to predict the movement of a ship, and in [20], the results of this paper are generalized to the standard $\mathcal{H}_\infty$ control problem with delays in the control input. Using the techniques of this paper, Koeman [13] solved the mixed sensitivity problem for general nonrational SISO systems.

The paper is organized as follows. Section II reviews the necessary machinery for our class of infinite-dimensional systems. In Section III we review stability properties and Smith predictors for dead-time systems. Section IV is about a solution of three simple $\mathcal{H}_\infty$ problems by using a Smith predictor parameterization of the controller. The three problems considered are $\mathcal{H}_\infty$ minimization of a weighted complementary sensitivity function, a weighted input sensitivity function, and a tracking/model-matching $\mathcal{H}_\infty$ problem. Section V treats the mixed sensitivity problem. The method is tested on two examples. Most of the proofs are in the Appendix.

II. PRELIMINARIES

The spectral norm of $M \in \mathbb{C}^{n\times n}$ is denoted as $\|M\|$. The spaces $\mathcal{H}_2$ and $\mathcal{H}_\infty$ denote the standard Hardy spaces defined on the open right-half plane. Their respective norms are interrelated through

$$\|G\|_{\mathcal{H}_\infty} = \sup_{u\in\mathcal{H}_2} \frac{\|Gu\|_{\mathcal{H}_2}}{\|u\|_{\mathcal{H}_2}}.$$

We normally do not mention dimensions and simply write $\mathcal{H}_\infty$ when we mean $\mathcal{H}_\infty^{n\times m}$. For $G \in \mathcal{H}_\infty$, the adjoint $G^\sim$ satisfies $G^\sim(s) = (G(-\overline{s}))^*$. The elements of $\mathcal{H}_\infty$ are called *stable*, and G is said to be *bistable* if $G, G^{-1} \in \mathcal{H}_\infty$. Somewhat less standard is the following class of transfer matrices.

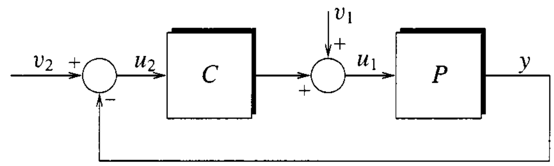

Fig. 1. A feedback configuration; setup for stability.

Definition 2.1: The quotient field of $\mathcal{H}_\infty$ is denoted by $\mathcal{F}_\infty$, that is

$$\mathcal{F}_\infty^{n\times m} := \left\{H^{-1}G : G \in \mathcal{H}_\infty^{n\times m},\ H \in \mathcal{H}_\infty^{n\times n},\ \det H \neq 0\right\}.$$

□

This field $\mathcal{F}_\infty$ will be our class of transfer matrices. An element $M \in \mathcal{F}_\infty$ is said to *strictly proper* if there is a $\rho \in \mathbb{R}$ such that

$$\lim_{s\to\infty,\, \mathrm{Re}\, s>\rho} \|M(s)\| = 0. \tag{2}$$

The dimension of a signal u is denoted with n_u, so, $u(s) \in \mathbb{C}^{n_u}$ for frequency domain signals.

Definition 2.2 (Closed-Loop Stability): Given $P, C \in \mathcal{F}_\infty$ the loop in Fig. 1 is *stable* if the four transfer matrices from v_1, v_2 to u_1 and u_2 are stable. In such cases, C is said to be *stabilizing* or is said to *stabilize the plant* P. □

A beautiful result by Smith [26], states that a plant $P \in \mathcal{F}_\infty$ is stabilizable by some $C \in \mathcal{F}_\infty$ if and only if matrices $P_n, P_d, \hat{X}, \hat{Y} \in \mathcal{H}_\infty$ exist such that

$$P = P_d^{-1}P_n \quad \text{and} \quad P_d\hat{X} + P_n\hat{Y} = I.$$

In such cases, $P = P_d^{-1}P_n$ is said to be a *strongly coprime factorization* of P, and $C := \hat{Y}\hat{X}^{-1}$ is then a stabilizing controller. This result is intriguing considering that $\mathcal{F}_\infty$ is a field in which not all elements have a strongly coprime factorization; that is, some plants $P \in \mathcal{F}_\infty$ are not stabilizable. se^{-s} is one such example [26]. As Smith showed, the stabilizable plants are exactly those plants that have a strongly coprime factorization, and this means that, as far as synthesis of stabilizing controllers is concerned, we may use the powerful tricks that come with strongly coprime factorizations. A central result in this respect is the following lemma.

Lemma 2.3: Let $P, C \in \mathcal{F}_\infty$, and suppose that

$$P = P_d^{-1}P_n, \qquad C = C_d^{-1}C_n \tag{3}$$

are strongly coprime factorizations over $\mathcal{H}_\infty$. Then, the closed loop of Fig. 1 is stable if and only if

$$\begin{bmatrix} C_d & -C_n \\ P_n & P_d \end{bmatrix} \tag{4}$$

is bistable. Conversely, if (4) is bistable, then (3) are strongly coprime factorizations and the closed loop is stable.

Proof: Given the result by Smith [26], this can be proved the same way as the rational case. ■

Not all elements of $\mathcal{F}_\infty$ have an implementation in time-domain as a causal operator. e^s being one such example. In most of the problems that we consider, the causality condition is an

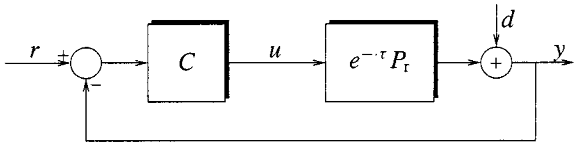

Fig. 2. A standard feedback configuration with a dead-time plant.

easy byproduct of other results, but on occasion we shall need the more abstract result of [36] that $C \in \mathcal{F}_\infty$ is causal if it is bounded on some right-half plane in the sense that

$$\exists\, \rho \in \mathbb{R} \qquad \text{such that} \sup_{s\in\mathbb{C},\, \mathrm{Re}\, s>\rho} \|C(s)\| < \infty. \tag{5}$$

III. Stability and Causality for Dead-Time Systems: Smith Predictors

In this section, we specialize the results on stability and causality to dead-time systems $e^{-s\tau}P_r(s)$, where P_r is some rational matrix.

Consider the feedback loop of Fig. 2. The plant $P(s) := e^{-s\tau}P_r(s)$ is a dead-time system with a delay τ. As a result, for any causal controller C the transfer matrix

$$T_{P,C} := (I+PC)^{-1}PC$$

from r to y will also exhibit a time-delay of at least τ. Therefore, $T_{P,C}$ can be expressed as

$$T_{P,C} = e^{-\cdot\tau}T_0, \tag{6}$$

where T_0 is some causal transfer matrix. The idea of the Smith predictor is, roughly, to design a controller C_0 for the rational part P_r of the plant so as to obtain some desired complementary sensitivity function

$$T_{P_r,C_0} := (I+P_rC_0)^{-1}P_rC_0$$

and then to solve C from the equality $T_{P,C} = e^{-\cdot\tau}T_{P_r,C_0}$. It is well known that this yields the controller, called the Smith predictor

$$C = \left(I + (1-e^{-\cdot\tau})C_0P_r\right)^{-1}C_0.$$

If P is stable, then it is not difficult to see that C stabilizes $e^{-\cdot\tau}P_r$ if and only if C_0 stabilizes the rational P_r. The Smith predictor, thus, has the important asset that it relegates the problem of closed-loop stability to that of a finite-dimensional system. If P_r is unstable, then a modified Smith predictor can be used much the same way, as follows.

Lemma 3.1 (Modified Smith Predictor): Consider the system depicted in Fig. 2 and assume that P_r is proper. Let F be a rational matrix such that $P_r(F - e^{-\cdot\tau}I)$ is stable. Then, the modified Smith predictor

$$\begin{aligned} C &:= \left(I + C_0P_r(F-e^{-\cdot\tau}I)\right)^{-1}C_0 \\ &= C_0\left(I + P_r(F-e^{-\cdot\tau}I)C_0\right)^{-1} \end{aligned} \tag{7}$$

stabilizes $P := e^{-\cdot\tau}P_r$ if and only if C_0 stabilizes P_rF.

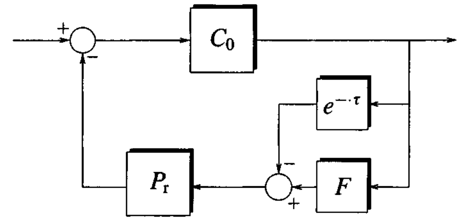

Fig. 3. The modified Smith predictor.

Furthermore, the various sensitivity functions are related as

$$\begin{aligned} S_{P,C} &:= (I+PC)^{-1} \\ &= S_{P_rF,C_0} + P_r(F-e^{-\cdot\tau}I)C_0S_{P_rF,C_0} \\ CS_{P,C} &= C_0S_{P_rF,C_0} \\ T_{P,C} &= e^{-\cdot\tau}P_rC_0S_{P_rF,C_0}. \end{aligned} \tag{8}$$

□

The proof is given shortly.

If P_r has only one unstable pole and if that pole is $s = 0$, then $P_r(I - e^{-\cdot\tau}I)$ is stable. Hence, we may take $F = I$, which renders the classical Smith predictor. For other unstable plants P_r, a rational F such that $P_r(F-e^{-\cdot\tau}I)$ is stable may be found through interpolation.

If C_0 is taken strictly proper, then the modified Smith predictor C is causal. This may be clear from the configuration in Fig. 3 and may be formally verified from (5). In particular, this shows that any dead-time system $e^{-\cdot\tau}P_r$ is stabilizable by causal control provided that P_r is proper, and goes the other way: if P_r is nonproper (and $\tau > 0$), then $e^{-\cdot\tau}P_r$ is not stabilizable by causal control, even if we allow for nonlinear and time-varying controllers. This follows from a time-domain argument: let v_1 be a step input (see Fig. 1). If P_r is nonproper then because of its differentiating action the output y is not defined, and feedback cannot instantaneously counteract the effect of the step input v_1 because of the delay in the loop.

The proof of Lemma 3.1 uses a result that we also need in the next sections.

Lemma 3.2: Suppose that P_r is proper rational and that $P_r = P_{r,d}^{-1}P_{r,n}$ is a rational strongly coprime factorization.

Then, $P_{r,d}^{-1}(e^{-\cdot\tau}P_{r,n})$ is a strongly coprime factorization of $e^{-\cdot\tau}P_r$, and $P_{r,d}^{-1}(P_{r,n}F)$ is a strongly coprime factorization of P_rF for any rational F for which $P_r(F-e^{-\cdot\tau}I)$ is stable.

Proof: It follows essentially from [26, remark following the proof of Theorem 1]. A concrete proof goes as follows.

Define Z as

$$Z = P_r(F - e^{-\cdot\tau}I),$$

and assume that F is such that Z is stable. From the identity

$$[\,P_{r,n}F \quad P_{r,d}\,] = [\,P_{r,d}Z + e^{-\cdot\tau}P_{r,n} \quad P_{r,d}\,] \tag{9}$$

it follows that $[P_{r,n}F \quad P_{r,d}]$ has full row rank in the closed right-half plane. It also has full row rank at infinity because $\det P_{r,d}(\infty) \neq 0$. Hence, the pair $(P_{r,d}, P_{r,n}F)$, being rational, is a strongly right coprime pair. So, $X, Y \in \mathcal{H}_\infty$ exist such that

$$P_{r,d}X + P_{r,n}FY = I.$$

From this, it can be seen that $(P_{r,d},\, e^{-\cdot\tau}P_{r,n})$ is a strongly right coprime pair as well because

$$\begin{aligned} &P_{r,d}(X+ZY)+e^{-\cdot\tau}P_{r,n}Y \\ &\quad = P_{r,d}X + P_{r,n}(F-e^{-\cdot\tau}I)Y + e^{-\cdot\tau}P_{r,n}Y = I \end{aligned}$$

and $X+ZY$ and Y are stable. ■

Proof of Lemma 3.1: Let $C_0 = C_{0,d}^{-1}C_{0,n}$ be a left coprime factorization of a controller C_0 that stabilizes P_rF. Then, by Lemma 2.3, the matrix

$$\begin{bmatrix} C_{0,d} & -C_{0,n} \\ P_{r,n}F & P_{r,d} \end{bmatrix}$$

is bistable. As a result

$$\begin{bmatrix} C_d & -C_n \\ e^{-\cdot\tau}P_{r,n} & P_{r,d} \end{bmatrix} := \begin{bmatrix} C_{0,d} & -C_{0,n} \\ P_{r,n}F & P_{r,d} \end{bmatrix} \cdot \begin{bmatrix} I & 0 \\ -P_r(F-e^{-\cdot\tau}I) & I \end{bmatrix} \tag{10}$$

is bistable as well, implying that $C_d^{-1}C_n$ stabilizes $e^{-\cdot\tau}P_r$. It is easy to see that $C_d^{-1}C_n$ is the modified Smith predictor (7).

Conversely, if $C = C_d^{-1}C_n$ is a strongly coprime factorization of a controller C that stabilizes $e^{-\cdot\tau}P_r$, then the left-hand side of (10) is bistable according to Lemma 2.3. Consequently

$$\begin{bmatrix} C_{0,d} - C_{0,n}P_{r,n}F & P_{r,d} \end{bmatrix} := \begin{bmatrix} C_d & -C_n \\ e^{-\cdot\tau}P_{r,n} & P_{r,d} \end{bmatrix} \cdot \begin{bmatrix} I & 0 \\ P_r(F-e^{-\cdot\tau}I) & I \end{bmatrix}$$

is then bistable as well, implying that $C_0 := C_{0,d}^{-1}C_{0,n}$ stabilizes P_rF. It is easy to verify that C_0 defined this way is the controller that satisfies (7). ■

IV. Smith Predictors for $\mathcal{H}_\infty$ Control

As noted, the Smith predictor—and the modified Smith predictor—relegate the stability problem to one of a finite-dimensional system. In this section, we show that the Smith predictor parameterization can also be used to transform certain $\mathcal{H}_\infty$ problems for dead-time systems into finite-dimensional $\mathcal{H}_\infty$ problems.

From Lemma 3.1, we copy the connections between the various sensitivity functions

$$\begin{aligned} S_{P,C} &= S_{P_rF,C_0} + P_r(F-e^{-\cdot\tau}I)C_0S_{P_rF,C_0} \\ CS_{P,C} &= C_0S_{P_rF,C_0} \\ T_{P,C} &= e^{-\cdot\tau}P_rC_0S_{P_rF,C_0}. \end{aligned}$$

Example 4.1 (Complementary Sensitivity Function): $\mathcal{H}_\infty$-norm minimization of the weighted complementary sensitivity function $W_2T_{P,C}$ of the dead-time system is equivalent to that of a rational system because

$$\begin{aligned} \|W_2T_{P,C}\|_{\mathcal{H}_\infty} &= \|W_2e^{-\cdot\tau}P_rC_0S_{P_rF,C_0}\|_{\mathcal{H}_\infty} \\ &= \|W_2P_rC_0S_{P_rF,C_0}\|_{\mathcal{H}_\infty}. \end{aligned}$$

□

Example 4.2 (Input Sensitivity Function): The input sensitivity function $CS_{P,C}$ in the dead-time system equals $C_0S_{P_rF,C_0}$. Hence, $\mathcal{H}_\infty$-norm minimization of the weighted $W_2CS_{P,C}$ for dead-time systems can be solved as a rational $\mathcal{H}_\infty$ optimization problem.

See [3] for a different solution. □

Incidentally, reducing the $\mathcal{H}_\infty$ norm of the input sensitivity function $W_2CS_{P,C}$ improves the stability robustness of the loop against uncertainties in the dead-time τ, provided that P_r is strictly proper; see [39].

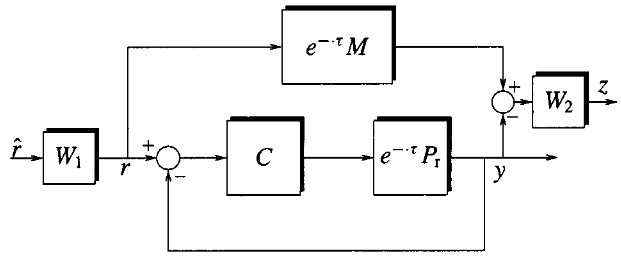

Fig. 4. Tracking/model-matching.

Example 4.3 (Tracking and Model Matching): Suppose we wish the output y to track a reference signal r (see Fig. 2). One way to optimize tracking is to minimize the $\mathcal{H}_\infty$-norm of the map

$$W_2\left(e^{-\cdot\tau}M - T_{P,C}\right)W_1$$

from $\hat{r}$ to z, as shown in Fig. 4. Here, W_1 and W_2 are some stable, rational weighting matrices and $e^{-\cdot\tau}M$ is a preferred map from r to y. It is advisable to include a time-delay of $e^{-s\tau}$ in this preferred map, as we did, because with a causal controller that is the shortest delay we can achieve in the map from r to y. With the controller parameterized as a modified Smith predictor, we get that

$$\begin{aligned} &\|W_2(e^{-\cdot\tau}M - T_{P,C})W_1\|_{\mathcal{H}_\infty} \\ &\quad = \|W_2(e^{-\cdot\tau}M - e^{-\cdot\tau}P_rC_0S_{P_rF,C_0})W_1\|_{\mathcal{H}_\infty} \\ &\quad = \|W_2(M - P_rC_0S_{P_rF,C_0})W_1\|_{\mathcal{H}_\infty}. \end{aligned}$$

The delay term has vanished, and the optimal controller can, hence, be obtained from a pure rational $\mathcal{H}_\infty$ optimization problem. □

The duality between the complementary sensitivity function T and sensitivity function S

$$S_{P,C} = T_{(1/P),(1/C)}$$

may suggest that minimization of $\|W_1S\|_{\mathcal{H}_\infty}$ is practically the same as that of $\|W_1T\|_{\mathcal{H}_\infty}$. This process in a way is true for rational systems, but when applied to dead-time systems the resulting controller turns out to be noncausal. The causality condition requires some more steps. A solution is given in [10, Sec. 4.2.3]. The problem is also a special case of the mixed sensitivity problem solved next.

V. The Mixed Sensitivity Problem

In this section, we minimize the $\mathcal{H}_\infty$-norm of

$$\begin{bmatrix} W_1(I+PC)^{-1} \\ W_2C(I+PC)^{-1} \end{bmatrix} \tag{11}$$

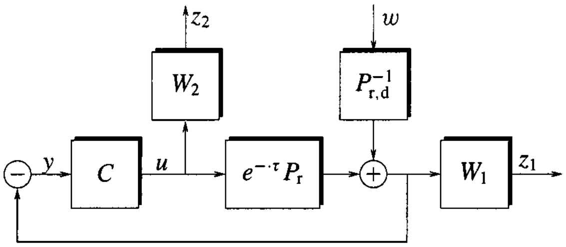

Fig. 5. The mixed sensitivity configuration.

with respect to stabilizing controllers $C \in \mathcal{F}_\infty$. This process is commonly called the *mixed sensitivity problem*, though that term is also used for $\mathcal{H}_\infty$-norm minimization of

$$\begin{bmatrix} W_1(I+PC)^{-1} \\ \hat{W}_2PC(I+PC)^{-1} \end{bmatrix}. \tag{12}$$

These two equations are essentially equivalent because we may absorb P into $\hat{W}_2$, but (11) has the advantage over (12) that P in (11) is allowed to have poles on the imaginary axis, whereas imaginary poles of P makes (12) violate the assumptions needed for the standard solution of the $\mathcal{H}_\infty$ control problem.

As before, the plant P is assumed of the form $e^{-\cdot\tau}P_r$ with P_r rational, and the weighting matrices W_1 and W_2 are assumed stable and rational.

Actually, we shall minimize the $\mathcal{H}_\infty$-norm of

$$H := \begin{bmatrix} W_1(I+PC)^{-1}P_{r,d}^{-1} \\ W_2C(I+PC)^{-1}P_{r,d}^{-1} \end{bmatrix} \tag{13}$$

where $P_{r,d}$ is a stable, rational matrix from a coprime factorization of the rational part of the plant

$$P_r = P_{r,d}^{-1}P_{r,n}. \tag{14}$$

If so desired, the factor $P_{r,d}$ can be chosen to be inner (i.e., $P_{r,d}^{\sim}P_{r,d} = I$ and $P_{r,d}$ stable) and then (11) and (13) have identical $\mathcal{H}_\infty$-norm. Also, in the SISO case, $P_{r,d}^{-1}$ could be absorbed into the other weights W_1 and W_2, but it has appeared useful in the rational case to keep $P_{r,d}^{-1}$ as a separate weight, see [15], and therefore we do it here as well.

Fig. 5 shows the closed-loop configuration corresponding to the mixed sensitivity problem, and H defined in (13) may be recognized as the transfer matrix from w to $\left[\begin{smallmatrix} z_1 \\ -z_2 \end{smallmatrix}\right]$. The open loop (i.e., with the controller taken away) can be expressed as a map from (u, y) to (w, z_1, z_2) as

$$\begin{bmatrix} -z_1 \\ z_2 \\ -w \end{bmatrix} = \begin{bmatrix} 0 & W_1 \\ W_2 & 0 \\ e^{-\cdot\tau}P_{r,n} & P_{r,d} \end{bmatrix} \begin{bmatrix} u \\ y \end{bmatrix}.$$

Theorem 5.1: Assume that P_r is proper, and let (14) be a rational coprime factorization of P_r over $\mathcal{H}_\infty$. Assume that W_1 and W_2 are stable rational weights. Define G as

$$G = \begin{bmatrix} 0 & W_1 \\ W_2 & 0 \\ e^{-s\tau}P_{r,n} & P_{r,d} \end{bmatrix} \in \mathcal{H}_\infty^{(n_{z_1}+n_{z_2}+n_w)\times(n_u+n_y)} \tag{15}$$

and suppose that G for zero time-delay has full column rank on the imaginary axis, including infinity. Let $\gamma > 0$. A stabilizing causal controller $C \in \mathcal{F}_\infty$ exists such that $\|H\|_{\mathcal{H}_\infty} < \gamma$ if and only if the following three conditions hold.

1) Define

$$J_\gamma := \begin{bmatrix} I_{n_{z_1}} & 0 & 0 \\ 0 & I_{n_{z_2}} & 0 \\ 0 & 0 & -\gamma^2 I_{n_w} \end{bmatrix}, \qquad \hat{J} := \begin{bmatrix} I_{n_u} & 0 \\ 0 & -I_{n_y} \end{bmatrix}. \tag{16}$$

$G^{\sim}J_\gamma G$ has the same number of positive and negative eigenvalues as $\hat{J}$ everywhere on the imaginary axis, and the eigenvalues of $G^{\sim}J_\gamma G$ are bounded away from zero on the imaginary axis.

2) A bistable Q exists such that $G^{\sim}J_\gamma G = Q^{\sim}\hat{J}Q$, and the lower-right $n_y \times n_y$-block M_{22} of $M := GQ^{-1}$ is bistable.

3) $\gamma > \|W_1(\infty)P_{r,d}^{-1}(\infty)\|$ if $\tau > 0$.

In this case, all stabilizing controllers C that achieve $\|H\|_{\mathcal{H}_\infty} < \gamma$ are parameterized by

$$C = (Z_{11}U + Z_{12})\,(Z_{21}U + Z_{22})^{-1} \tag{17}$$

where $Z := Q^{-1} \in \mathcal{H}_\infty^{(n_u+n_y)\times(n_u+n_y)}$ and $U \in \mathcal{H}_\infty^{n_u\times n_y}$ with $\|U\|_{\mathcal{H}_\infty} < 1$. Moreover, Q (which is not unique) can be chosen such that its upper-right $n_u \times n_y$ block Q_{12} satisfies $\lim_{\mathrm{Re}\, s>0,\, s\to\infty} Q_{12}(s) = 0$, and for that choice of Q, the controller (17) is causal for any strictly proper U.

Proof: Most of the proof is technical and can be found in the Appendix. Here, we only prove the necessity of Condition 3) because we need it in the rest of this section.

Consider Fig. 5. We see that in open loop $W_1(\infty)P_{r,d}^{-1}(\infty)$ is the direct feed-through term from w to z_1. If $\tau > 0$, then the map from w to z_1 on the time interval $(0, \epsilon)$ is not affected by feedback for $\epsilon < \tau$, and the $\mathcal{L}_2$-gain on this interval $(0, \epsilon)$ approaches $\|W_1(\infty)P_{r,d}^{-1}(\infty)\|$ as $\epsilon \to 0$. Hence, $\|H\|_{\mathcal{H}_\infty} \geq \|W_1(\infty)P_{r,d}^{-1}(\infty)\|$ for any causal controller. ■

The proof in the Appendix shows a clear interpretation of the three items: Condition 1) holds if and only if $\|H\|_{\mathcal{L}_\infty} < \gamma$ can be achieved by some controller, possibly not stabilizing and noncausal; Condition 2) characterizes when this can be done with a stabilizing controller, and Condition 3) expresses when, in addition, C can taken to be causal. For the delay-free case, Condition 3) is void, and Conditions 1) and 2) are then the well-known frequency domain conditions of Green *et al.* [12].

Theorem 5.1 in its present form is not very practical because it is generally difficult to find a bistable Q that solves the nonrational spectral factorization problem asked for in Condition 2)

$$G^{\sim}J_\gamma G = Q^{\sim}\hat{J}Q, \quad Q \text{ bistable}. \tag{18}$$

Owing to the specific structure of G, however, the nonrational part can be removed from the spectral factorization problem, leaving a pure rational factorization problem to be solved. This can be done as follows.

Given γ, let Π be the rational part of $G^{\sim}J_\gamma G$ (that is, $G^{\sim}J_\gamma G$ for zero time-delay)

$$\Pi := \begin{bmatrix} W_2^{\sim}W_2 - \gamma^2 P_{r,n}^{\sim}P_{r,n} & -\gamma^2 P_{r,n}^{\sim}P_{r,d} \\ -\gamma^2 P_{r,d}^{\sim}P_{r,n} & W_1^{\sim}W_1 - \gamma^2 P_{r,d}^{\sim}P_{r,d} \end{bmatrix}.$$

It is easy to verify that with the time-delays included we have that

$$G^{\sim}J_\gamma G = \begin{bmatrix} \Pi_{11} & e^{s\tau}\Pi_{12} \\ e^{-s\tau}\Pi_{21} & \Pi_{22} \end{bmatrix}. \tag{19}$$

If $\tau = 0$, then this is a rational problem and we can proceed with Theorem 5.2. The interesting case is when $\tau > 0$. By Theorem 5.1, Condition 3) we have that, then

$$\Pi_{22}(\infty) := W_1^{\sim}(\infty)W_1(\infty) - \gamma^2 P_{r,d}^{\sim}(\infty)P_{r,d}(\infty) < 0.$$

Therefore, Π_{22}^{-1} exists and is proper. Now write $e^{-s\tau}\Pi_{22}^{-1}\Pi_{21}$ as a sum of a nonrational, but stable, part, F_{stab}, and a proper rational part, R

$$e^{-s\tau}\Pi_{22}^{-1}\Pi_{21} = F_{\text{stab}} + R; \quad F_{\text{stab}} \in \mathcal{H}_\infty \quad \text{and } R \text{ rational.} \tag{20}$$

This result is always possible.[1] Then, we have that

$$\Theta := \begin{bmatrix} I & -F_{\text{stab}}^{\sim} \\ 0 & I \end{bmatrix} G^{\sim} J_\gamma G \begin{bmatrix} I & 0 \\ -F_{\text{stab}} & I \end{bmatrix} = \begin{bmatrix} \Pi_{11} - \Pi_{12}\Pi_{22}^{-1}\Pi_{21} + R^{\sim}\Pi_{22}R & R^{\sim}\Pi_{22} \\ \Pi_{22}R & \Pi_{22} \end{bmatrix}. \tag{21}$$

The point here is that Θ defined here is rational and proper, and that the factor $\left[\begin{smallmatrix} I & 0 \\ -F_{\text{stab}} & I \end{smallmatrix}\right]$ that we extracted is bistable. Therefore, finding a nonrational spectral factor Q in (18) is equivalent to finding a rational spectral factor Q_r such that

$$\Theta = Q_r^{\sim}\hat{J}Q_r, \qquad Q_r \text{ bistable,} \tag{22}$$

and the spectral factors Q and Q_r are related by

$$Q = Q_r \begin{bmatrix} I & 0 \\ F_{\text{stab}} & I \end{bmatrix}.$$

Theorem 5.1 can now be phrased a bit more concretely.

Theorem 5.2: Assume that P_r is proper, and let (14) be a rational coprime factorization of P_r over $\mathcal{H}_\infty$. Assume that W_1 and W_2 are stable rational weights. Define G as in (15), and suppose that for zero time-delay G has full column rank on the imaginary axis, including infinity. Let $\gamma > 0$. A stabilizing causal controller exists such that $\|H\|_{\mathcal{H}_\infty} < \gamma$ if and only if the following three conditions hold.

1) $\Theta(\infty)$ is nonsingular and has the same number of positive and negative eigenvalues as $\hat{J}$, and $\Theta(j\omega)$ is nonsingular on the imaginary axis.
2) A bistable Q_r exists such that $\Theta = Q_r^{\sim}\hat{J}Q_r$, and the lower-right $n_y \times n_y$-block M_{22} of $M := G\left[\begin{smallmatrix} I & 0 \\ -F_{\text{stab}} & I \end{smallmatrix}\right]Q_r^{-1}$ is bistable.
3) $\Theta_{22}(\infty) < 0$ if $\tau > 0$.

In this case, all stabilizing controllers C that achieve $\|H\|_{\mathcal{H}_\infty} < \gamma$ are parameterized by

$$C = (Z_{11}U + Z_{12})\,(Z_{21}U + Z_{22})^{-1}, \tag{23}$$

where $Z := \left[\begin{smallmatrix} I & 0 \\ -F_{\text{stab}} & I \end{smallmatrix}\right]Q_r^{-1} \in \mathcal{H}_\infty^{(n_u+n_y)\times(n_u+n_y)}$ and $U \in \mathcal{H}_\infty^{n_u \times n_y}$ with $\|U\|_{\mathcal{H}_\infty} < 1$. Moreover, Q_r (which is not unique) can be chosen such that its upper-right $n_u \times n_y$ block $Q_{r,12}$ satisfies $Q_{r,12}(\infty) = 0$, and for this choice of Q_r, the controller is causal for any strictly proper U.

[1] We construct one in Theorem 5.3. For the SISO case with simple poles only, one can be found from a partial fraction expansion

$$\frac{\Pi_{12}(s)}{\Pi_{22}(s)} = a_0 + \sum_{k=1}^{n} a_k/(s-\mu_k):$$

let $F_{\text{stab}}(s) := a_0(e^{-s}-1) + \sum_{k=1}^{n} a_k(e^{-s}-e^{-\mu_k})/(s-\mu_k)$ and $R(s) := a_0 + \sum_{k=1}^{n} a_k e^{-\mu_k}/(s-\mu_k)$.

Proof: The conditions are item by item equivalent to those of Theorem 5.1. [Note that in Condition 1) the nonsingularity of $\Theta(j\omega)$ implies that the number of negative eigenvalues and the number of positive eigenvalues of $\Theta(j\omega)$ is equal to those of $\hat{J}$ for *every* ω.] ■

This formulation makes the problem suited for computation. Given γ, a spectral factor Q_r can be computed using the solution of an associated Riccati equation [2], [12]; the stable factor F_{stab} can be constructed from a partial fraction expansion and bistability of M_{22} can then be tested with help of the Nyquist criterion. For the delay-free case, the stability test is much simpler. In such cases, M_{22} is bistable if and only if the solution of the Riccati equation is positive semidefinite [12]. For our nonrational system, it is still possible to obtain necessary and sufficient conditions in terms of Riccati equations, but the conditions are necessarily more involved. This is analogous to the Lyapunov type conditions in [24].

Theorem 5.3: Assume that P_r is proper, and let (14) be a rational coprime factorization of P_r over $\mathcal{H}_\infty$. Assume that W_1 and W_2 are stable rational weights. Form a realization of the rational part of G

$$\begin{bmatrix} 0 & W_1(s) \\ W_2(s) & 0 \\ P_{r,n}(s) & P_{r,d}(s) \end{bmatrix} = C(sI-A)^{-1}B + D = C(sI-A)^{-1}[B_1 \;\; B_2] + [D_1 \;\; D_2]$$

where the partitioning of $B = [B_1 \;\; B_2]$ and $D = [D_1 \;\; D_2]$ corresponds to the partitioning of G. Assume that A has all its eigenvalues in the open left-half plane and that $C(sI-A)^{-1}B + D$ has full column rank on the imaginary axis and at ∞.

A stabilizing causal controller exists such that $\|H\|_{\mathcal{H}_\infty} < \gamma$ if and only if the following three conditions hold.

1) $D^T J_\gamma D$ is nonsingular and has the same number of positive and negative eigenvalues as $\hat{J}$, and the Hamiltonian $\mathcal{H}_\gamma$ defined as

$$\mathcal{H}_\gamma = \begin{bmatrix} A & 0 \\ -C^T J_\gamma C & -A^T \end{bmatrix} - \begin{bmatrix} L_1 \\ L_2 \end{bmatrix} \left(D^T J_\gamma D\right)^{-1} [-L_2^T \quad L_1^T]$$

has no imaginary eigenvalues. Here

$$\begin{bmatrix} L_1 \\ L_2 \end{bmatrix} = \begin{bmatrix} B \\ -C^T J_\gamma D \end{bmatrix} + (e^{-\tau A_H} - I) \times \begin{bmatrix} B_1 - B_2\hat{D}_{22}^{-1}\hat{D}_{21} & 0_{n\times n_y} \\ -C^T J_\gamma(D_1 - D_2\hat{D}_{22}^{-1}\hat{D}_{21}) & 0_{n\times n_y} \end{bmatrix}$$

in which

$$A_H := \begin{bmatrix} A & 0 \\ -C^T J_\gamma C & -A^T \end{bmatrix} - \begin{bmatrix} B_2 \\ -C^T J_\gamma D_2 \end{bmatrix} \hat{D}_{22}^{-1} [D_2^T J_\gamma C \quad B_2^T]$$

$$\begin{bmatrix} \hat{D}_{11} & \hat{D}_{12} \\ \hat{D}_{21} & \hat{D}_{22} \end{bmatrix} := D^T J_\gamma D.$$

2) The Hamiltonian $\mathcal{H}_\lambda \in \mathbb{R}^{2n\times 2n}$ defined in Condition 1) has an n-dimensional stable eigenspace $\text{Im}\left[\begin{smallmatrix} X_1(\lambda) \\ X_2(\lambda) \end{smallmatrix}\right]$, [with

$X_i(\lambda) \in \mathbb{R}^{n\times n}$], and $X_1(\lambda) \in \mathbb{R}^{n\times n}$ is nonsingular for all $\lambda \geq \gamma$.

3) $\hat{D}_{22} < 0$.

If the above conditions hold, then nonsingular matrices Y and S exist such that $\hat{D}_{22} = -Y^T Y$ and $\hat{D}_{11} - \hat{D}_{12}\hat{D}_{22}^{-1}\hat{D}_{21} = S^T S$, then

$$Q_{r,\infty} := \begin{bmatrix} S & 0 \\ -Y^{-T}\hat{D}_{21} & Y \end{bmatrix}$$

satisfies $D^T J_\gamma D = Q_\infty^T \hat{J} Q_\infty$, and for Q_r and F_{stab}, we may take

$$Q_r \stackrel{s}{=} \left[\begin{array}{c|c} A & L_1 \\ \hline \hat{J}Q_\infty^{-T}\left(D^T J_\gamma C + B^T X_2(\gamma)X_1^{-1}(\gamma)\right) & Q_\infty \end{array}\right] \tag{24}$$

and

$$\begin{aligned} &F_{\text{stab}}(s) \\ &\quad = \hat{D}_{22}^{-1}\left[D_2^T J_\gamma C \quad B_2^T\right](sI - A_H)^{-1}\left(e^{-s\tau}I - e^{-\tau A_H}\right) \\ &\qquad \times \begin{bmatrix} B_1 - B_2\hat{D}_{22}^{-1}\hat{D}_{21} \\ -C^T J_\gamma\left(D_1 - D_2\hat{D}_{22}^{-1}\hat{D}_{21}\right) \end{bmatrix} \\ &\qquad + \left(e^{-s\tau} - 1\right)\hat{D}_{22}^{-1}\hat{D}_{21}. \end{aligned}$$

For this choice of Q_r, the upper-right block $Q_{r,12}$ of Q_r satisfies $Q_{r,12}(\infty) = 0$, and with Z defined as $Z = \left[\begin{smallmatrix} I & 0 \\ -F_{\text{stab}} & I \end{smallmatrix}\right] Q_r^{-1} \in \mathcal{H}_\infty^{(n_u+n_y)\times(n_u+n_y)}$, the controllers C defined as

$$C = (Z_{11}U + Z_{12})(Z_{21}U + Z_{22})^{-1} \tag{25}$$

are causal for any strictly proper U. Moreover, C stabilizes and achieves $\|H\|_{\mathcal{H}_\infty} < \gamma$ if and only if U is stable and $\|U\|_{\mathcal{H}_\infty} < 1$.

Proof: See the Appendix. ∎

It is well known that, given Condition 1), the Condition 2) can alternatively be expressed as the "stabilizing" solution $X(\lambda) := X_2(\lambda)X_1^{-1}(\lambda)$ of the Riccati equation $[X(\lambda) \quad -I]\mathcal{H}_\lambda\left[\begin{smallmatrix} I \\ X(\lambda) \end{smallmatrix}\right] = 0$ exists for all $\lambda \geq \gamma$. For zero time-delay, the factor $(e^{-\tau A_H} - I)$ is zero, and then the Riccati equation recovers the standard Riccati equation associated with this type spectral factorization problem [2], [12], and for such cases, Condition 2) holds if and only if the single test holds that $X(\gamma) \geq 0$ [12]. As it stands, with the time-delays, Condition 2) is more involved, but it is easy enough to allow to compute controllers.

Remark 1: The condition of Theorem 5.1, Condition 3), that

$$\gamma > \left\|W_1(\infty)P_{r,d}^{-1}(\infty)\right\| \tag{26}$$

is for the delay-free case not needed for the computation of controllers, and is indeed not necessarily true for stabilizing causal controllers to exist that make $\|H\|_{\mathcal{H}_\infty} < \gamma$. The condition (26) expresses therefore a gap between zero time-delay and nonzero time-delay, no matter how small the delay is. In practical cases, however, the inequality (26) will also hold for the delay-free case, for if C or P_r is strictly proper, then

$$\begin{aligned} \|H\|_{\mathcal{H}_\infty} &\geq \left\|W_1(I + PC)^{-1}P_{r,d}^{-1}\right\|_{\mathcal{H}_\infty} \\ &\geq \left\|W_1(\infty)\left(I + P(\infty)C(\infty)\right)^{-1}P_{r,d}^{-1}(\infty)\right\| \\ &= \left\|W_1(\infty)P_{r,d}^{-1}(\infty)\right\|. \end{aligned}$$

It may be shown that the causality condition (26) is connected to the essential spectral radius of a certain Hankel plus Toeplitz operator as considered in, e.g., [23] and [37].

Remark 2: The results of this section remain valid if everywhere in this section we replace the delay function $m(s) = e^{-\tau s}$ with an arbitrary inner function $m(s)$ for which mP_r is strictly proper. In that case, the matrix exponential $e^{-\tau A_H}$ in Theorem 5.3 should be replaced with the matrix function $m(A_H)$ (similar inner matrix functions are employed in [16]).

Remark 3: Another straightforward generalization of the results in this section is to replace the single time-delay $m(s) = e^{-\tau s}$ with a multiple time-delay $M(s) = \text{diag}(e^{-\tau_1 s}, \cdots, e^{-\tau_n s})$. This way $M(s)$ does not automatically commute with other operators. Going through the manipulations of this section will show that Theorems 5.1 and 5.2 remain valid, provided we assume the following:

1) $P = MP_r$, that is, the multiple delays occur in the components of the output;
2) M commutes with $P_{r,d}^{\sim}$ and $P_{r,d}^{\sim}P_{r,d}$;
3) M commutes with $W_1^{\sim}W_1$.

The second condition is not very stringent if P_r is stable, because then we may take $P_{r,d} = I$. Also, the third condition is not very stringend, because we may always choose $W_1(s)$ to be diagonal.

A. Controller Structure and Implementation

The most obvious choice for U in the controller parameterization C defined in (25) is $U = 0$. The resulting controller

$$C = Z_{12}Z_{22}^{-1}$$

is commonly called the central controller. With the explicit formulas for Q_r and F_{stab}, it is interesting to work out the structure of the central and other controllers and to see if and how such controllers can be implemented.

It is not difficult to verify that the general controller

$$C = (Z_{11}U + Z_{12})\,(Z_{21}U + Z_{22})^{-1}$$

can be rearranged as shown in Fig. 6(a), in which

$$M = \begin{bmatrix} -Z_{r,22}^{-1}Z_{r,21} & Z_{r,22}^{-1} \\ Z_{r,11} - Z_{r,12}Z_{r,22}^{-1}Z_{r,21} & Z_{r,12}Z_{r,22}^{-1} \end{bmatrix}$$

with

$$\begin{bmatrix} Z_{r,11} & Z_{r,12} \\ Z_{r,21} & Z_{r,22} \end{bmatrix} := Q_r^{-1} \in \mathcal{H}_\infty^{(n_u+n_y)\times(n_u+n_y)}.$$

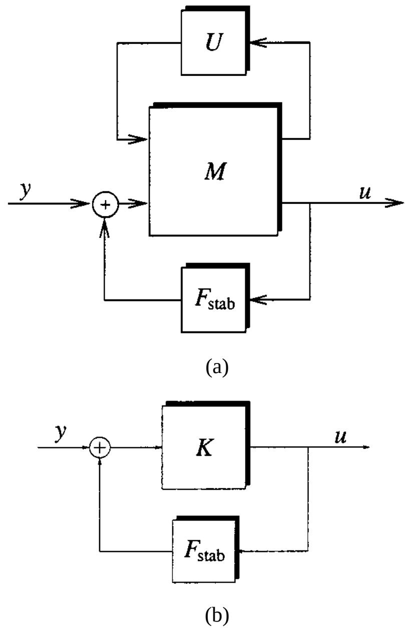

Fig. 6. (a) The controller as an LFT and (b) the central controller.

We are free to choose U (as long as it is stable and $\|U\|_{\mathcal{H}_\infty} < 1$) and $F_{\rm stab}$ is the infinite-dimensional part of the controller. Normally, we would choose $U = 0$ and then the controller simplifies to $C = (I - KF_{\rm stab})^{-1}K$, shown in Fig. 6(b), with $K := Z_{r,12}Z_{r,22}^{-1}$. This is the central controller. In the central controller, the block K is finite dimensional. The only infinite-dimensional part if $F_{\rm stab}$. The $F_{\rm stab}$ constructed in Theorem 5.3 is of the form

$$F_{\rm stab}(s) = \tilde{C}(sI - A_H)^{-1}\left(e^{-s\tau}I - e^{-\tau A_H}\right)\tilde{B} + (e^{-s\tau} - 1)\tilde{D} \tag{27}$$

and the fortunate implication of this is that its inverse Laplace transform (its impulse response) has compact support (see [8] and [11], for similar observations). In fact the impulse response of (27) is

$$f(t) := \begin{cases} -\tilde{C}e^{(t-\tau)A_H}\tilde{B} & \\ \quad + (\delta(t-\tau) - \delta(t))\tilde{D}, & \text{if } t \in [0^-, \tau^+) \\ 0, & \text{elsewhere} \end{cases}$$

and it has support equal to the dead-time τ. It is because of its finite support property that $F_{\rm stab}$ may be implemented without too much difficulty, certainly for simulation purposes. Normally, P_r is strictly proper and it may be verified that in such cases $\tilde{D} = 0$ and so the Dirac pulses in the impulse response $f(t)$ are then not present.

The McMillan degree of M equals the McMillan degree of Z_r, which in turn equals the sum of McMillan degrees of W_1, W_2, and P_r. This is in accordance with the delay free case, and it suggests that controllers of significantly lower order generically will not exist. In contrast, the description of $F_{\rm stab}$ as given in Theorem 5.3 often appears to contain hidden modes.

Remark: The construction of Q_r in Theorem 5.3 is such that $Z_{r,22}$ is biproper, so all of the inverses taken above are well defined. The construction in Theorem 5.3 further guarantees that $Z_{r,12}$ is strictly proper, and for that reason, the controller is causal for every strictly proper U.

B. Examples

We apply the method of the previous section to two examples. The first example is from Toker and Özbay [32], and the second is based on an example by Partington and Glover [24]. The MATLAB macros used for the computations are based on the results from the previous section.[2] The macros work as long as the delay is not too large. For large delays, different techniques have to be used; see [40]. In [19], one more example is reported.

Example 5.4: Reference [32] considered minimizing the $\mathcal{H}_\infty$-norm of

$$H_T := \begin{bmatrix} W_1(I+PC)^{-1} \\ \hat{W}_2PC(I+PC)^{-1} \end{bmatrix}$$

with

$$\begin{aligned} P(s) &= e^{-0.2s}\frac{1}{s-1} \\ W_1(s) &= 2\,\frac{s+1}{10s+1} \\ \hat{W}_2(s) &= 0.2(s+1.1). \end{aligned}$$

This problem is not yet in our form (13), but if we let

$$\begin{aligned} W_2(s) &= \hat{W}_2P(s)e^{0.2s}\,\frac{s-1}{s+1} = 0.2\,\frac{s+1.1}{s+1} \\ P_{r,d}(s) &= \frac{s-1}{s+1} \end{aligned}$$

then their H_T and our H

$$H = \begin{bmatrix} W_1(I+PC)^{-1}P_{r,d}^{-1} \\ W_2C(I+PC)^{-1}P_{r,d}^{-1} \end{bmatrix}$$

differ only a by unitary factor and, hence, have the same ∞-norm (note that $e^{-s\tau}$ and $P_{r,d}$ have modulus 1 on the imaginary axis). For this data, the rational part of the matrix G defined in (15) has McMillan degree 3, and so the matrix $X_1(\gamma)$ of Theorem 5.3 is a 3×3 matrix. Fig. 7 shows the smallest singular value of $X_1(\gamma)$ as a function of γ. At $\gamma = 0.6819$, the X_1 is singular for the first time as γ comes down from $+\infty$, and, therefore, that is the optimal γ. The central controller for γ slightly larger than optimal, $\gamma = \gamma_{\rm opt} + 0.0001$, is $C = (I - KF_{\rm stab})^{-1}K$, with

$$K(s) = \frac{4.6971s^2 + 5.6971s + 1}{0.000\,016s^3 + 1.4414s^2 + 1.4792s + 0.0379}$$

and with $F_{\rm stab}$ a system with impulse response as depicted in Fig. 8. The impulse response on its support is almost indistinguishable from a straight line. As with the delay-free case, some coefficients in K approach zero as γ approaches the optimal γ. Removing those terms from the controller description leaves a controller of reduced order, and this is normally the optimal controller. Our theory does not apply, however, to the optimal case

[2]The macros are available from http://www.math.utwente.nl/~meinsma/.

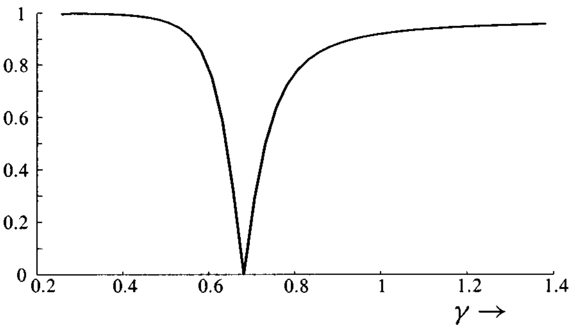

Fig. 7. The smallest singular value of $X_1(\gamma)$; Example 5.4.

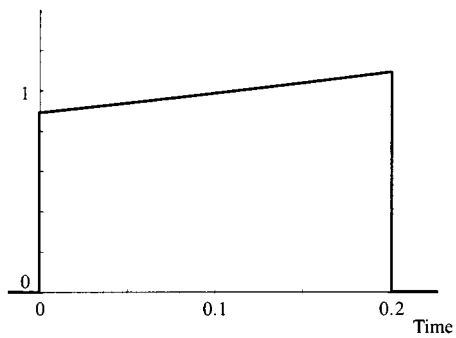

Fig. 8. Minus the impulse response of $F_{\rm stab}$; Example 5.4.

and care should thus be taken with setting some coefficients to zero, no matter how small they are. A separate Nyquist plot can be used to test for closed-loop stability for the reduced-order controller. □

Example 5.5: This example is taken from [24, Sec. 5]. We consider the dead-time system $P(s) := e^{-s\tau}/s$, and we want to find a controller that not only stabilizes P, but also stabilizes all plants in a neighborhood

$$\mathcal{P}_\epsilon := \left\{ \frac{P_n + \Delta_n}{P_d + \Delta_d} \colon \| [\Delta_n \quad \Delta_d] \|_{\mathcal{H}_\infty} < \epsilon \right\}.$$

Here, $P = P_d^{-1} P_n$ is strongly coprime factorization over $\mathcal{H}_\infty$ normalized in the sense that $|P_n(j\omega)|^2 + |P_d(j\omega)|^2 = 1$ for all frequencies. For our plant, $P(s) = e^{-s\tau}/s$ that is the case for

$$P_n(s) := e^{-s\tau}\frac{1}{s+1}; \qquad P_d(s) := \frac{s}{s+1}.$$

It is well known that, given C, the "stability radius" ϵ is determined as

$$\begin{aligned}\frac{1}{\epsilon} &= \left\| \begin{bmatrix} C \\ I \end{bmatrix} (I+PC)^{-1} [P \quad I] \right\|_{\mathcal{H}_\infty} \\ &= \left\| \begin{bmatrix} C(I+PC)^{-1}P_d^{-1} \\ (I+PC)^{-1}P_d^{-1} \end{bmatrix} \right\|_{\mathcal{H}_\infty}.\end{aligned}$$

Maximizing the stability radius over the stabilizing controllers is thus an $\mathcal{H}_\infty$-norm minimization problem. In accordance with [24], we find for $\tau = 1$ a maximal ϵ of $\epsilon = 1/\gamma = 0.4859$. A near optimal controller is $C = (I - KF_{\rm stab})^{-1}K$ with

$$K(s) = \frac{1}{0.000009s + 0.5561}$$

and with $F_{\rm stab}$ a system with impulse response

$$f(t) = \begin{cases} -1.3091 \cos\left(\sqrt{0.3091}\,(t-\tau)\right), & \text{if } t \in [0, \tau), \\ 0, & \text{otherwise.} \end{cases}$$

□

APPENDIX

A. Frequency Domain Proofs

In this subsection, we prove Theorem 5.1. This proof also proves Theorem 5.2 as the two theorems are equivalent.

Lemma 6.1 (Small Gain): Let M_{21}, M_{22}, B, and A be stable transfer matrices of appropriate dimensions, and suppose that M_{22} and A are invertible in $\mathcal{L}_\infty$ and that $\|M_{22}^{-1}M_{21}\|_{\mathcal{L}_\infty} \le 1$ and $\|BA^{-1}\|_{\mathcal{L}_\infty} < 1$. Then, $M_{21}B + M_{22}A$ is bistable if and only if M_{22} and A are bistable.

Proof: We make use of [5, Lemma 8.3], which states that if $\|U\|_{\mathcal{L}_\infty} < 1$ then U is stable if and only if $(I+U)^{-1}$ is stable.

Suppose M_{22} and A are bistable, and define $U := M_{22}^{-1}M_{21}BA^{-1}$. Because U is stable and $\|U\|_{\mathcal{H}_\infty} \le \|M_{22}^{-1}M_{21}\|_{\mathcal{H}_\infty}\|BA^{-1}\|_{\mathcal{H}_\infty} < 1$, we have that

$$(M_{21}B + M_{22}A)^{-1} = A^{-1}(U+I)^{-1}M_{22}^{-1}$$

is stable.

Conversely, if $(M_{21}B + M_{22}A)^{-1}$ is stable, then

$$(I + M_{22}^{-1}M_{21}BA^{-1})^{-1} = A(M_{21}B + M_{22}A)^{-1}M_{22}$$

is also stable, and because $\|M_{22}^{-1}M_{21}BA^{-1}\|_{\mathcal{L}_\infty} < 1$, we have that $M_{22}^{-1}M_{21}BA^{-1}$ is stable as well. Now, it is easy to see that

$$\begin{aligned}A^{-1} &= A^{-1}\left(I + M_{22}^{-1}M_{21}BA^{-1}\right)^{-1}\left(I + M_{22}^{-1}M_{21}BA^{-1}\right) \\ &= (M_{22}A + M_{21}B)^{-1}M_{22}\left(I + M_{22}^{-1}M_{21}BA^{-1}\right),\end{aligned}$$

and therefore A^{-1} is stable. Similarly, M_{22}^{-1} can be seen to stable. This completes the proof. (The rational case is proved in [34, pp. 274–275].) ■

As before, we use the short hands

$$J_\gamma := \begin{bmatrix} I_{n_{z_1}} & 0 & 0 \\ 0 & I_{n_{z_2}} & 0 \\ 0 & 0 & -\gamma^2 I_{n_w} \end{bmatrix}, \quad \hat{J} := \begin{bmatrix} I_{n_u} & 0 \\ 0 & -I_{n_y} \end{bmatrix}.$$

It is important to note that $n_y = n_w$, i.e., that J_γ and $\hat{J}$ have the same number of negative eigenvalues. The following theorem was proved for the rational case in [12].

Theorem 6.2: Let $M \in \mathcal{H}_\infty$, and suppose that $M^\sim J_\gamma M = \hat{J}$ almost everywhere on $j\mathbb{R}$, with J_γ and $\hat{J}$ as defined above. Consider the equality

$$\begin{bmatrix} H_1 \\ H_2 \end{bmatrix} = M \begin{bmatrix} U_1 \\ U_2 \end{bmatrix}$$

with $H_1 \in \mathcal{H}_\infty^{(n_{z_1}+n_{z_2})\times n_y}$, $H_2 \in \mathcal{H}_\infty^{n_w \times n_y}$, $U_1 \in \mathcal{H}_\infty^{n_u\times n_y}$, and $U_2 \in \mathcal{H}_\infty^{n_y \times n_y}$. Then, the following conditions are equivalent.

1) H_2 is bistable and $\|H_1 H_2^{-1}\|_{\mathcal{H}_\infty} < \gamma$.
2) M_{22} and U_2 are bistable and $\|U_1 U_2^{-1}\|_{\mathcal{H}_\infty} < 1$.

Proof: We use the fact that a transfer matrix $H_2 \in \mathcal{L}_\infty$ is invertible in $\mathcal{L}_\infty$ if and only if $H_2^\sim H_2 > \alpha I$ on the imaginary axis for some $\alpha > 0$

$$\begin{aligned}
&H_2 \text{ bistable, } \|H_1 H_2^{-1}\|_{\mathcal{H}_\infty} < \gamma \\
&\quad\Leftrightarrow H_2 \text{ bistable, } \left(H_1 H_2^{-1}\right)^\sim H_1 H_2^{-1} - \gamma^2 I < -\epsilon I < 0 \\
&\qquad \text{on the imaginary axis, for some } \epsilon > 0 \\
&\quad\Leftrightarrow H_2 \text{ bistable, } H_1^\sim H_1 - \gamma^2 H_2^\sim H_2 < -\delta I < 0 \\
&\qquad \text{on the imaginary axis, for some } \delta > 0 \\
&\quad\Leftrightarrow H_2 \text{ bistable, } \begin{bmatrix} H_1 \\ H_2 \end{bmatrix}^\sim J_\gamma \begin{bmatrix} H_1 \\ H_2 \end{bmatrix} < -\delta I < 0 \\
&\qquad \text{on the imaginary axis, for some } \delta > 0 \\
&\quad\Leftrightarrow H_2 \text{ bistable, } \begin{bmatrix} U_1 \\ U_2 \end{bmatrix}^\sim \hat{J} \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} < -\delta I < 0 \\
&\qquad \text{on the imaginary axis, for some } \delta > 0 \\
&\quad\Leftrightarrow H_2 \text{ bistable, } U_1^\sim U_1 - U_2^\sim U_2 < -\delta I < 0 \\
&\qquad \text{on the imaginary axis, for some } \delta > 0 \\
&\quad\Leftrightarrow M_{21} U_1 + M_{22} U_2 \text{ bistable, } U_2 \text{ is invertible in } \mathcal{L}_\infty \\
&\qquad \text{and } \|U_1 U_2^{-1}\|_{\mathcal{L}_\infty} < 1.
\end{aligned} \tag{28}$$

Next, we show that M_{22} is invertible and that $\|M_{22}^{-1} M_{21}\|_{\mathcal{L}_\infty} < 1$. By the fact that $M^\sim J_\gamma M = \hat{J}$, it follows that $M_{12}^\sim M_{12} - \gamma^2 M_{22}^\sim M_{22} = -I$. In particular, M_{22} is invertible in $\mathcal{L}_\infty$. Now, consider the equation

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

with $u_1, u_2, y_1, y_2 \in \mathcal{L}_2$. As $M^\sim J_\gamma M = \hat{J}$, there holds that

$$\|y_1\|_2^2 - \gamma^2 \|y_2\|_2^2 = \|u_1\|_2^2 - \|u_2\|_2^2.$$

For the specific case of $u_2 = -M_{22}^{-1} M_{21} u_1$, the output y_2 is identically zero, and the above equality reduces to

$$\|y_1\|_2^2 = \|u_1\|_2^2 - \|M_{22}^{-1} M_{21} u_1\|_2^2.$$

As $\|y_1\|_2^2 \geq 0$ and u_1 is arbitrary, we necessarily have that $\|M_{22}^{-1} M_{21}\|_{\mathcal{L}_\infty} \leq 1$. Using this case and Lemma 6.1, the condition of (28) is equivalent to that $\|U_1 U_2^{-1}\|_{\mathcal{L}_\infty} < 1$ and M_{22} and U_2 are bistable. ■

In the remainder of the section, we prove Theorem 5.2 item by item. Extensive use is made of the strongly coprime factorizations $C = \hat{C}_n \hat{C}_d^{-1}$ and $P = P_{r,d}^{-1}(e^{-\cdot\tau} P_{r,n})$. Then, according to [26], a controller $C \in \mathcal{F}$ stabilizes $P \in \mathcal{F}$ if and only if $\chi := P_{r,d}\hat{C}_d + e^{-\cdot\tau} P_{r,n} \hat{C}_n$ is bistable. We also make use of the following expression:

$$\begin{aligned}
\begin{bmatrix} H \\ I \end{bmatrix} &= \begin{bmatrix} W_1 (I+PC)^{-1} P_{r,d}^{-1} \\ W_2 C (I+PC)^{-1} P_{r,d}^{-1} \\ I \end{bmatrix} \\
&= \begin{bmatrix} W_1 \Big(P_{r,d}^{-1} \left[P_{r,d}\hat{C}_d + e^{-\cdot\tau} P_{r,n}\hat{C}_n\right] \hat{C}_d^{-1}\Big)^{-1} P_{r,d}^{-1} \\ W_2 C \Big(P_{r,d}^{-1} \left[P_{r,d}\hat{C}_d + e^{-\cdot\tau} P_{r,n}\hat{C}_n\right] \hat{C}_d^{-1}\Big)^{-1} P_{r,d}^{-1} \\ I \end{bmatrix} \\
&= \begin{bmatrix} W_1 \hat{C}_d \\ W_2 \hat{C}_n \\ \chi \end{bmatrix} \chi^{-1} \\
\chi &:= P_{r,d}\hat{C}_d + e^{-\cdot\tau} P_{r,n}\hat{C}_n \\
&= \begin{bmatrix} 0 & W_1 \\ W_2 & 0 \\ e^{-\cdot\tau} P_{r,n} & P_{r,d} \end{bmatrix} \begin{bmatrix} \hat{C}_n \\ \hat{C}_d \end{bmatrix} \chi^{-1} \\
\chi &:= P_{r,d}\hat{C}_d + e^{-\cdot\tau} P_{r,n}\hat{C}_n \\
&= G \begin{bmatrix} \hat{C}_n \\ \hat{C}_d \end{bmatrix} \chi^{-1} \\
\chi &:= P_{r,d}\hat{C}_d + e^{-\cdot\tau} P_{r,n}\hat{C}_n.
\end{aligned} \tag{29}$$

The matrix χ is the closed-loop "characteristic polynomial," and closed-loop stability is equivalent to χ being bistable [26].

Lemma 6.3: Let $\gamma > 0$ be given. The condition of Theorem 5.1, Condition 1) is necessary for the existence of a stabilizing $C \in \mathcal{F}_\infty$ such that $\|H\|_{\mathcal{H}_\infty} < \gamma$.

Proof: Suppose $C \in \mathcal{F}_\infty$ stabilizes and achieves $\|H\|_{\mathcal{H}_\infty} < \gamma$. As C is assumed to stabilize P, we have that C has a factorization over $\mathcal{H}_\infty$ such that $C = \hat{C}_n \hat{C}_d^{-1}$ with

$$\chi := P_{r,d}\hat{C}_d + e^{-\cdot\tau} P_{r,n}\hat{C}_n = I.$$

Let $P = \hat{P}_n \hat{P}_d^{-1}$ be a right coprime factorization of P. Using the fact that $\chi = I$ and that the bottom row-block of G is $[e^{-\cdot\tau} P_{r,n} \quad P_{r,d}]$, we get that

$$G \begin{bmatrix} \hat{C}_n & \hat{P}_d \\ \hat{C}_d & -\hat{P}_n \end{bmatrix} = \begin{bmatrix} H & Y \\ I & 0 \end{bmatrix} \tag{30}$$

with H the closed-loop transfer matrix, and Y whatever comes out. In what follows, $\sigma_{\min}$ denotes the smallest singular value. By assumption, $\sigma_{\min} G(j\omega)$ is bounded away from zero and because $\left[\begin{smallmatrix} \hat{C}_n & \hat{P}_d \\ \hat{C}_d & -\hat{P}_n \end{smallmatrix}\right]$ is bistable, it follows from (30) that $\sigma_{\min} Y(j\omega)$ is also bounded away from zero as a function of ω. Consider the identity

$$\begin{aligned}
&\begin{bmatrix} \hat{C}_n & \hat{P}_d \\ \hat{C}_d & -\hat{P}_n \end{bmatrix}^\sim G^\sim J_\gamma G \begin{bmatrix} \hat{C}_n & \hat{P}_d \\ \hat{C}_d & -\hat{P}_n \end{bmatrix} \\
&\quad = \begin{bmatrix} H^\sim H - \gamma^2 I & H^\sim Y \\ Y^\sim H & Y^\sim Y \end{bmatrix}.
\end{aligned} \tag{31}$$

It now follows from a standard Schur complement result that $G^\sim J_\gamma G$ has at any $s = j\omega$ at least as many negative eigenvalues as $H^{\sim}H - \gamma^2 I$ and at least as many positive eigenvalues as $Y^{\sim}Y$. This accounts in fact for all of the eigenvalues so that $G^{\sim}J_\gamma G$ has indeed the same inertia as $\hat{J}$ everywhere on the imaginary axis. None of these eigenvalues as a function of $s = j\omega$ can approach zero because $\|H\|_{\mathcal{H}_\infty}$ is strictly less than γ, and $Y \in \mathcal{H}_\infty$ and $\sigma_{\min} Y(j\omega)$ is bounded away from zero. ■

Lemma 6.4: Let $\gamma > 0$ be given, and assume Theorem 5.1, Condition 1), holds. The condition of Theorem 5.1, Condition 2), is necessary and sufficient for the existence of a stabilizing $C \in \mathcal{F}_\infty$ such that $\|H\|_{\mathcal{H}_\infty} < \gamma$.

In particular, under these conditions, C stabilizes and achieves $\|H\|_{\mathcal{H}_\infty} < \gamma$ if and only if $C = (Z_{11}U + Z_{12})(Z_{21}U + Z_{22})^{-1}$ for some stable U with $\|U\|_{\mathcal{H}_\infty} < 1$.

Proof—Necessity: Suppose that $C \in \mathcal{F}_\infty$ stabilizes and achieves $\|H\|_{\mathcal{H}_\infty} < \gamma$. From Lemma 5.1, Condition 1), we have that $G^{\sim}J_\gamma G$ has the same number of positive and negative eigenvalues as $\hat{J}$. Then, also, the rational $\Theta(j\omega)$ defined in (21) has this number of positive and negative eigenvalues, including at $\omega = \infty$. Then, according to [18, Cor. 3.1], Θ has a spectral factorization

$$\Theta = Q_r^{\sim}\hat{J}Q_r, \qquad Q_r \text{ bistable} \tag{32}$$

if and only if Θ has no *equalizing vectors*. An equalizing vector of Θ is a nonzero $u \in \mathcal{H}_2$ such that $\Theta u \in \mathcal{H}_2^\perp$. Suppose, to obtain a contradiction, that such equalizing vectors u exist, then $\hat{u} := \left[\begin{smallmatrix} I & 0 \\ F_{\text{stab}} & I\end{smallmatrix}\right]u$ is an equalizing vector of $G^{\sim}J_\gamma G$. Partition $v := G\hat{u}$ as $v = \left[\begin{smallmatrix} v_1 \\ v_2 \end{smallmatrix}\right]$. Then, on the one hand, we have that

$$\begin{aligned}\|v_1\|_2^2 - \gamma^2\|v_2\|_2^2 &= \langle v,\, J_\gamma v\rangle = \langle G\hat{u},\, J_\gamma G\hat{u}\rangle \\ &= \langle \hat{u},\, \underbrace{G^{\sim}J_\gamma G\hat{u}}_{\in \mathcal{H}_2^\perp}\rangle = 0\end{aligned}$$

so that

$$\|v_1\|_2 = \gamma\|v_2\|_2 \tag{33}$$

whereas on the other hand, we have that

$$\begin{aligned}H^{\sim}v_1 - \gamma^2 v_2 &= [\,H^{\sim} \quad I\,]\, J_\gamma G\hat{u} \\ &= \chi^{-\sim}\,[\,\hat{C}_n^{\sim} \quad \hat{C}_d^{\sim}\,]\, \underbrace{G^{\sim}J_\gamma G\hat{u}}_{\in \mathcal{H}_2^\perp} \in \mathcal{H}_2^\perp.\end{aligned}$$

Because $v_2 \in \mathcal{H}_2$, we have that $v_2 = (1/\gamma^2)\pi_+(H^{\sim}v_1)$ with π_+ denoting the orthogonal projection from $\mathcal{L}_2$ onto $\mathcal{H}_2$. Consequently

$$\gamma^2\|v_2\|_2 = \|\pi_+(H^{\sim}v_1)\|_2 \le \|H^{\sim}v_1\|_2 < \gamma\|v_1\|_2. \tag{34}$$

This equation contradicts (33); hence, no equalizing vectors u exist, and as a result, (32) has a bistable solution Q_r. Now, $Q := Q_r\left[\begin{smallmatrix} I & 0 \\ F_{\text{stab}} & I\end{smallmatrix}\right]$ is a bistable spectral factor of $G^{\sim}J_\gamma G$.

It remains to show that the lower right $n_y \times n_y$ block of GQ^{-1} is bistable. As Q is bistable, we can write any stable factorization $C = \hat{C}_n\hat{C}_d^{-1}$ of C as

$$\begin{bmatrix} \hat{C}_n \\ \hat{C}_d \end{bmatrix} = Q^{-1}\begin{bmatrix} U_1 \\ U_2 \end{bmatrix} \tag{35}$$

for some stable U_1 and U_2. From (29), we see that

$$\begin{bmatrix} H\chi \\ \chi \end{bmatrix} = G\begin{bmatrix} \hat{C}_n \\ \hat{C}_d \end{bmatrix} = GQ^{-1}\begin{bmatrix} U_1 \\ U_2 \end{bmatrix}.$$

Note that $M := GQ^{-1}$ satisfies $M^{\sim}J_\gamma M = \hat{J}$ and that the controller is stabilizing if and only if χ is bistable. Hence, by Theorem 6.2, the controller (35) is stabilizing and achieves $\|H\|_{\mathcal{H}_\infty} < \gamma$ if and only if M_{22} and U_2 is bistable and $\|U_1U_2^{-1}\|_{\mathcal{H}_\infty} < 1$.

Sufficiency: In particular, we see that such controllers exist if and only if M_{22} is bistable: simply take $U_1 = 0$ and $U_2 = I$. It is easy to see that (35) is just another way of writing that $C = (Z_{11}U + Z_{12}U)(Z_{21}U + Z_{22})^{-1}$, where $Z := Q^{-1}$ and $U := U_1U_2^{-1}$. ■

Lemma 6.5: Let $\gamma > 0$ be given, and assume Theorem 5.1, Conditions 1) and 2), hold. The condition of Theorem 5.1, Condition 3), is then necessary and sufficient for the existence of a causal stabilizing $C \in \mathcal{F}_\infty$ such that $\|H\|_{\mathcal{H}_\infty} < \gamma$.

Concretely, under these conditions, a rational spectral factor Q_r (as defined in Theorem 5.2) exists such that $Q_{r,12}(\infty) = 0$ and then $C = (Z_{11}U + Z_{12})(Z_{21}U + Z_{22})^{-1}$ is causal for every strictly proper U.

Proof: The necessity of Condition 3) was shown in the proof of Theorem 5.1. The sufficiency can be seen by explicitly writing a spectral factor.

Condition 3) implies that $\Theta_{22}(\infty) < 0$. Then, Theorem 5.2, Condition 2), implies that the constant matrix $\Theta(\infty)$ can be decomposed as

$$\begin{aligned}&\Theta(\infty) = S^T\hat{J}S, \\ &\quad S = \begin{bmatrix} S_{11} & 0 \\ S_{21} & S_{22} \end{bmatrix} \in \mathbb{R}^{(n_u+n_y)\times(n_u+n_y)}, \qquad \det S_{22} \neq 0.\end{aligned}$$

For any spectral factor $\tilde{Q}_r$ of Θ, the matrix $Q_r := S\tilde{Q}_r^{-1}(\infty)\tilde{Q}_r$ is also a bistable spectral factor of Θ and it is normalized in that

$$Q_r(\infty) = S = \begin{bmatrix} S_{11} & 0 \\ S_{21} & S_{22} \end{bmatrix}.$$

Then, also, $Z := (Q_r\left[\begin{smallmatrix} I & 0 \\ F_{\text{stab}} & I\end{smallmatrix}\right])^{-1}$ is block lower triangular at infinity in the sense that

$$\lim_{s\to\infty,\, \operatorname{Re} s > 0} Z(s) = \begin{bmatrix} ? & 0 \\ ? & S_{22}^{-1} \end{bmatrix}.$$

Now, let U be strictly proper. Then, a $\rho \in \mathbb{R}$ exists such that

$$C := (Z_{11}U + Z_{12})(Z_{21}U + Z_{22})^{-1}$$

satisfies

$$\lim_{s\to\infty,\,\mathrm{Re}\,s>\rho} C(s) = \lim_{s\to\infty,\,\mathrm{Re}\,s>\rho} Z_{12}(s)Z_{22}^{-1}(s) = S_{12}S_{22}^{-1} = 0.$$

Hence, C is strictly proper, and in particular, we see that C is causal. ■

B. Riccati Equations

In this subsection, we prove Theorem 5.3. Whenever we write

$$Y \stackrel{s}{=} \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] \tag{36}$$

we mean that A is square and that $Y(s) = C(sI-A)^{-1}B+D$. Usually, A, B, C, and D are constant matrices, and $Y(s) = C(sI-A)^{-1}B+D$ is then called a realization of Y. In this subsection, however, the matrices B, C, and D may depend on s corresponding to a nonrational Y. The "realization" (36) is then essentially a Rosenbrock system matrix description [25]. First, some preliminary results.

Lemma 6.6: Let

$$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$$

be a realization of the rational part of G, and then the rational part of $G^\sim J_\gamma G$ has the realization

$$\left[\begin{array}{c|cc} \hat{A} & \hat{B}_1 & \hat{B}_2 \\ \hline \hat{C}_1 & \hat{D}_{11} & \hat{D}_{12} \\ \hat{C}_2 & \hat{D}_{21} & \hat{D}_{22} \end{array}\right] := \left[\begin{array}{cc|c} A & 0 & B \\ -C^TJ_\gamma C & -A^T & -C^TJ_\gamma D \\ \hline D^TJ_\gamma C & B^T & D^TJ_\gamma D \end{array}\right]. \tag{37}$$

Here, the partitionings are compatible with that of $\hat{J}$. Then, Θ as defined in (21) has the realization as shown in (38) at the bottom of the page, where A_H is the Hamiltonian matrix $A_H := \hat{A} - \hat{B}_2\hat{D}_{22}^{-1}\hat{C}_2$. Moreover

$$F_{\mathrm{stab}} \stackrel{s}{=} \left[\begin{array}{c|c} A_H & (e^{-s\tau}I - e^{-\tau A_H})(\hat{B}_1 - \hat{B}_2\hat{D}_{22}^{-1}\hat{D}_{21}) \\ \hline \hat{D}_{22}^{-1}\hat{C}_2 & (e^{-s\tau}-1)\hat{D}_{22}^{-1}\hat{D}_{21} \end{array}\right].$$

Proof: Let Π be the rational part of $G^\sim J_\gamma G$. It is easy to verify that the realization in (37) is a realization of Π. Including the time-delays, we obtain the "realization"

$$G^\sim J_\gamma G = \begin{bmatrix} \Pi_{11} & e^{s\tau}\Pi_{12} \\ e^{-s\tau}\Pi_{21} & \Pi_{22} \end{bmatrix} \stackrel{s}{=} \left[\begin{array}{c|cc} \hat{A} & e^{-s\tau}\hat{B}_1 & \hat{B}_2 \\ \hline e^{s\tau}\hat{C}_1 & \hat{D}_{11} & e^{s\tau}\hat{D}_{12} \\ \hat{C}_2 & e^{-s\tau}\hat{D}_{21} & \hat{D}_{22} \end{array}\right].$$

We associate with $G^\sim J_\gamma G$ the equation $\left[\begin{smallmatrix} y_1 \\ y_2 \end{smallmatrix}\right] = G^\sim J_\gamma G \left[\begin{smallmatrix} u_1 \\ u_2 \end{smallmatrix}\right]$. This equation can be rearranged as follows:

$$\begin{bmatrix} y_1 \\ -u_2 \end{bmatrix} = \underbrace{\begin{bmatrix} \Pi_{11} - \Pi_{12}\Pi_{22}^{-1}\Pi_{21} & e^{s\tau}\Pi_{12}\Pi_{22}^{-1} \\ e^{-s\tau}\Pi_{22}^{-1}\Pi_{21} & -\Pi_{22}^{-1} \end{bmatrix}}_{\Omega} \begin{bmatrix} u_1 \\ y_2 \end{bmatrix}. \tag{39}$$

This function defines Ω. Rearranging the realization of $G^\sim J_\gamma G$ similarly gives a realization of Ω as shown in (39a) at the bottom of the next page. Note that the A-matrix here is A_H. Looking at the lower left block of Ω, we see that

$$e^{-s\tau}\Pi_{22}^{-1}\Pi_{21} \stackrel{s}{=} \left[\begin{array}{c|c} A_H & e^{-s\tau}\left(\hat{B}_1 - \hat{B}_2\hat{D}_{22}^{-1}\hat{D}_{21}\right) \\ \hline \hat{D}_{22}^{-1}\hat{C}_2 & e^{-s\tau}\hat{D}_{22}^{-1}\hat{D}_{21} \end{array}\right].$$

It is this term that we have to write as $F_{\mathrm{stab}} + R$ with F_{stab} in $\mathcal{H}_\infty$ and R a rational matrix. The claim is that this holds for

$$F_{\mathrm{stab}} \stackrel{s}{=} \left[\begin{array}{c|c} A_H & (e^{-s\tau}I - e^{-\tau A_H})\left(\hat{B}_1 - \hat{B}_2\hat{D}_{22}^{-1}\hat{D}_{21}\right) \\ \hline \hat{D}_{22}^{-1}\hat{C}_2 & (e^{-s\tau}-1)\hat{D}_{22}^{-1}\hat{D}_{22} \end{array}\right]$$

$$R \stackrel{s}{=} \left[\begin{array}{c|c} A_H & e^{-\tau A_H}\left(\hat{B}_1 - \hat{B}_2\hat{D}_{22}^{-1}\hat{D}_{21}\right) \\ \hline \hat{D}_{22}^{-1}\hat{C}_2 & \hat{D}_{22}^{-1}\hat{D}_{21} \end{array}\right].$$

That the sum equals $e^{-s\tau}\Pi_{22}^{-1}\Pi_{21}$ is trivial, that R is rational is also trivial, and that F_{stab} is in $\mathcal{H}_\infty$ follows from the fact that the poles of $(sI-A_H)^{-1}$ cancel against the term $(e^{-s\tau}I - e^{-\tau A_H})$

$$\begin{aligned} &(sI-A_H)^{-1}\left(e^{-s\tau}I - e^{-\tau A_H}\right) \\ &\quad = e^{-s\tau}(sI-A_H)^{-1}\left(I - e^{\tau(sI-A_H)}\right) \\ &\quad = e^{-s\tau}(sI-A_H)^{-1}\sum_{k=1}^{\infty} -\frac{\tau^k}{k!}(sI-A_H)^k \\ &\quad = e^{-s\tau}\sum_{k=1}^{\infty} -\frac{\tau^k}{k!}(sI-A_H)^{k-1}. \end{aligned}$$

$$\left[\begin{array}{c|cc} \hat{A} & e^{-\tau A_H}\hat{B}_1 + (I - e^{-\tau A_H})\hat{B}_2\hat{D}_{22}^{-1}\hat{D}_{21} & \hat{B}_2 \\ \hline \hat{C}_1 e^{\tau A_H} + \hat{D}_{12}\hat{D}_{22}^{-1}\hat{C}_2(I - e^{\tau A_H}) & \hat{D}_{11} & \hat{D}_{12} \\ \hat{C}_2 & \hat{D}_{21} & \hat{D}_{22} \end{array}\right] \tag{38}$$

By the symmetry property, we have that $e^{s\tau}\Pi_{12}\Pi_{22}^{-1} = F_{\rm stab}^{\sim} + R^{\sim}$. Because we have from (39) that

$$e^{s\tau}\Pi_{12}\Pi_{22}^{-1} \stackrel{s}{=} \left[\begin{array}{c|c} A_H & \hat{B}_2\hat{D}_{22}^{-1} \\ \hline e^{s\tau}\left(\hat{C}_1 - \hat{D}_{12}\hat{D}_{22}^{-1}\hat{C}_2\right) & e^{s\tau}\hat{D}_{12}\hat{D}_{22}^{-1} \end{array}\right]$$

it is reasonable to guess—and indeed true—that the $F_{\rm stab}$ and R constructed earlier also satisfy (39b) and (39c), at the bottom of the page.

Based on this, we now combine the various blocks and form the realization as shown in (40) at the bottom of the page.

As a final step, we associate with (40) the equation

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} \Pi_{11} - \Pi_{12}\Pi_{22}^{-1}\Pi_{21} & R^{\sim} \\ R & -\Pi_{22}^{-1} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

and we rewrite it as

$$\begin{bmatrix} y_1 \\ u_2 \end{bmatrix} = \underbrace{\begin{bmatrix} \Pi_{11} - \Pi_{12}\Pi_{22}^{-1}\Pi_{21} + R^{\sim}\Pi_{22}R & R^{\sim}\Pi_{22} \\ \Pi_{22}R & \Pi_{22} \end{bmatrix}}_{\Theta} \cdot \begin{bmatrix} u_1 \\ -y_2 \end{bmatrix}.$$

Here, we recognize Θ as defined in (21). In terms of state-space manipulations, we similarly obtain (38), which is what we needed to show. ■

Proof of Theorem 5.3: We prove the equivalence with Theorem 5.2 item by item.

The conditions imposed on G are the same as those in Theorem 5.2 because A is assumed to have no imaginary eigenvalues.

Condition 1) First of all, note that $\Theta(\infty) = D^T J_\gamma D$.

Looking at the realization (38) of Θ, we see that the Hamiltonian $\mathcal{H}_\gamma$ defined in Theorem 5.2, Condition 1) is the "A-matrix" of the realization of the *inverse* of Θ. A standard Schur complement gives that

$$\det(\hat{A} - j\omega I)\det(\Theta(j\omega)) = \det(D^T J_\gamma D)\det(\mathcal{H}_\gamma - j\omega I)$$

with $\hat{A}$ as defined in (37). By assumption, A has no eigenvalues on the imaginary axis; hence, $\hat{A}$ has not either. The conclusion is that $\Theta(j\omega)$ is nonsingular everywhere on the imaginary axis, including infinity, if and only if $D^T J_\gamma D$ is nonsingular and $\mathcal{H}_\gamma$ has no imaginary eigenvalues.

Condition 2) By the canonical factorization theorem [2], [18], the condition that $X_1(\gamma)$ is nonsingular is equivalent to the existence of a bistable spectral factor Q_r. In Condition 2) that $X_1(\lambda)$ exists and is nonsingular for all $\lambda \geq \gamma$ is equivalent to that M_{22} is bistable. This can be seen as follows. If the solution $X_1(\lambda)$ does not exist or is singular for some $\lambda \geq \gamma$, then by canonical factorization theorem no bistable Q_r can exist, and, hence, by Theorem 5.1, λ-optimal controllers do not exist. If $X_1(\lambda)$ *does* exist and is nonsingular for all $\lambda \geq \gamma$, then $X := X_2X_1^{-1}$ as a function of λ is continuous. Then, also, $M := G\left[\begin{smallmatrix} I & 0 \\ -F_{\rm stab} & I \end{smallmatrix}\right]Q_r^{-1}$ depends continuously on λ and, so, also does the Nyquist plot[3] of M_{22}. For λ large enough, M_{22} is bistable, if now for $\lambda = \gamma$, the M_{22} would not be bistable, then by continuity the Nyquist plot for some λ in between will have to be zero at some $s = j\omega$. That is impossible because—despite whether M_{22} is bistable—we have that $M_{12}^*(j\omega)M_{12}(j\omega) - \lambda^2 M_{22}^*(j\omega)M_{22}(j\omega) = -I$ everywhere on the imaginary axis

[3] M_{22} is bistable if and only if the Nyquist plot $\det M_{22}(j\mathbb{R})$ does not encircle the origin. The Nyquist criterion assumes that $\lim_{\mathrm{Re}\, s \geq 0, |s| \to \infty} \det M_{22}(s)$ exists. It may be verified that for our choice of Q_r [namely, $Q_r(\infty)$ being block lower triangular] implies this limit to exist with nonzero limit $\det(P_{r,d}(\infty)Q_{r,22}^{-1}(\infty))$.

$$\Omega \stackrel{s}{=} \left[\begin{array}{c|cc} \hat{A} - \hat{B}_2\hat{D}_{22}^{-1}\hat{C}_2 & e^{-s\tau}\left(\hat{B}_1 - \hat{B}_2\hat{D}_{22}^{-1}\hat{D}_{21}\right) & \hat{B}_2\hat{D}_{22}^{-1} \\ \hline e^{s\tau}\left(\hat{C}_1 - \hat{D}_{12}\hat{D}_{22}^{-1}\hat{C}_2\right) & \hat{D}_{11} - \hat{D}_{12}\hat{D}_{22}^{-1}\hat{D}_{21} & e^{s\tau}\hat{D}_{12}\hat{D}_{22}^{-1} \\ \hat{D}_{22}^{-1}\hat{C}_2 & e^{-s\tau}\hat{D}_{22}^{-1}\hat{D}_{21} & -\hat{D}_{22}^{-1} \end{array}\right] \tag{39a}$$

$$F_{\rm stab}^{\sim} \stackrel{s}{=} \left[\begin{array}{c|c} A_H & \hat{B}_2\hat{D}_{22}^{-1} \\ \hline \left(\hat{C}_1 - \hat{D}_{12}\hat{D}_{22}^{-1}\hat{C}_2\right)\left(e^{s\tau}I - e^{\tau A_H}\right) & (e^{s\tau} - 1)\hat{D}_{12}\hat{D}_{22}^{-1} \end{array}\right] \tag{39b}$$

$$R^{\sim} \stackrel{s}{=} \left[\begin{array}{c|c} A_H & \hat{B}_2\hat{D}_{22}^{-1} \\ \hline \left(\hat{C}_1 - \hat{D}_{12}\hat{D}_{22}^{-1}\hat{C}_2\right)e^{\tau A_H} & \hat{D}_{12}\hat{D}_{22}^{-1} \end{array}\right] \tag{39c}$$

$$\begin{bmatrix} \Pi_{11} - \Pi_{12}\Pi_{22}^{-1}\Pi_{21} & R^{\sim} \\ R & -\Pi_{22}^{-1} \end{bmatrix} \stackrel{s}{=} \left[\begin{array}{c|cc} A_H & e^{-\tau A_H}\left(\hat{B}_1 - \hat{B}_2\hat{D}_{22}^{-1}\hat{D}_{21}\right) & \hat{B}_2\hat{D}_{22}^{-1} \\ \hline \left(\hat{C}_1 - \hat{D}_{12}\hat{D}_{22}^{-1}\hat{C}_2\right)e^{\tau A_H} & \hat{D}_{11} - \hat{D}_{12}\hat{D}_{22}^{-1}\hat{D}_{21} & \hat{D}_{12}\hat{D}_{22}^{-1} \\ \hat{D}_{22}^{-1}\hat{C}_2 & \hat{D}_{22}^{-1}\hat{D}_{21} & -\hat{D}_{22}^{-1} \end{array}\right] \tag{40}$$

for every λ. Hence, no such singular points ω exist, and M_{22} is bistable for $\lambda = \gamma$.

Condition 3) $\hat{D}_{22} = \Theta_{22}(\infty)$.

It is a matter of manipulation to check that $X_2 X_1^{-1}$ is the stabilizing solution of a Riccati equation, and that the realization of Q_r indeed defines a bistable spectral factor of Θ. (See, for example, the similar derivation in [18].) The F_{stab} was already determined in Lemma 6.6.

That C is causal for strictly proper U is shown in Lemma 6.5. ■

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