Model Uncertainty and Robust Stability for Multivariable Systems

ELEC 571L – Robust Multivariable Control
prepared by: Greg Stewart
• Representing model uncertainty.

• Common configurations of unstructured uncertainty.

• Generalizing uncertainty: the $M\Delta$-structure.

• Robust stability.
Representing Model Uncertainty
• One common convention is to use complex transfer matrix perturbations.

• For example, additive uncertainty is given by:

\[
G_p(s) = G(s) + \Delta_a(s)
\]

where the stable transfer matrix \(\Delta_a(s)\) has the same dimension as the nominal model \(G(s)\)
The assumption of completely unstructured uncertainty is represented by a bound on the maximum singular value:

\[ \bar{\sigma}(\Delta(j\omega)) \leq 1 \quad \text{for all } \omega \]

We are assuming even less knowledge about this perturbation than we did for the SISO case.

This transfer matrix may have any internal structure at all.
• The assumption of stability and the bound

\[ \bar{\sigma}(\Delta(j\omega)) \leq 1 \]

• permits any of the following for \( \Delta(s) \):
  – Zero matrices,
  – Complex matrices,
  – Real matrices,
  – Diagonal matrices,
  – Ill-conditioned matrices,
  – Well-conditioned matrices,
  – Full matrices,
  – Sparse matrices,
  – etc.
• Similar to the SISO case, the model uncertainty will often be shaped with known transfer matrices,

\[
\begin{align*}
G(s) & \xrightarrow{\Delta_a(s)} W_2(s) \\
W_1(s) & \xrightarrow{} G_p(s)
\end{align*}
\]

• Gain of \( W_2 \) and \( W_1 \) should be large for uncertain frequencies and directions of the model.
The bound on the largest singular value

\[ \sigma(\Delta(j\omega)) \leq 1 \quad \text{for all } \omega \]

is equivalent to bounding the \( H_\infty \) norm

\[ \left\| \Delta(s) \right\|_\infty \leq 1 \]

and is commonly used as notation.
Common Configurations of Model Uncertainty
Three examples

- **Additive uncertainty**

![Additive uncertainty diagram]

- **Multiplicative input uncertainty**

![Multiplicative input uncertainty diagram]

- **Multiplicative output uncertainty**

![Multiplicative output uncertainty diagram]
• In MIMO systems, it is more difficult to move uncertainty around than in SISO systems.

• Consider for example, the three previous cases:

\[ G_p = G + E_A \] (additive)
\[ G_p = G (I + E_I) \] (multiplicative input)
\[ G_p = (I + E_O) G \] (multiplicative output)
The Importance of Configuration

- Suppose that we have $G_p = G (I + E_i)$, then the size of the uncertainty is $\| E_i \|_\infty$.

- Now, suppose we wish to look at uncertainty in terms of $E_0$.

- The two multiplicative configurations may be made equivalent by writing

  $$E_0 G = G E_i$$

- Then (ignoring issues of invertibility),

  $$E_0 = G E_i G^{-1}$$
The Importance of Configuration

- Then taking the maximum singular value of both sides,

\[ \overline{\sigma}(E_O) = \overline{\sigma}(G \cdot E_I \cdot G^{-1}) \]

- If we assume that the perturbations are unstructured, then we can “pull-out”

\[ \overline{\sigma}(E_O) = \overline{\sigma}(E_I) \cdot \overline{\sigma}(G) \cdot \overline{\sigma}(G^{-1}) \]

\[ = \overline{\sigma}(E_I) \cdot \frac{\overline{\sigma}(G)}{\sigma(G)} \]
If the matrix $G$ is ill-conditioned, then $\sigma(G)$ is small, then

$$\bar{\sigma}(E_O) = \bar{\sigma}(E_I) \cdot \frac{\sigma(G)}{\sigma(G)}$$

and therefore $\bar{\sigma}(E_O)$ must be large!

A similar effect would have happened if we had tried to represent a $E_O$ perturbation in terms of $E_I$. 
• We can represent *both* input and output multiplicative uncertainty easily in terms of additive uncertainty.

• First, we equate the perturbations,

\[ E_A = E_O G = G E_I \]
Then, for unstructured matrices \( \{E_A, E_O, E_I\} \), we find:

\[
\bar{\sigma}(E_A) = \bar{\sigma}(E_O) \bar{\sigma}(G) = \bar{\sigma}(G) \bar{\sigma}(E_I)
\]

and since we only deal with the maximum singular value (i.e. not the minimum), then we are insensitive to ill-conditioned matrices \( G \).

Those of you doing the CD control project should take note of this result!
Generalizing Uncertainty: the $\Delta$-Structure
Consider a typical MIMO feedback loop with multiplicative output uncertainty:

\[
\begin{align*}
|\Delta_0(s)| & \leq 1 \\
\end{align*}
\]

the unstructured, stable transfer matrix \( \Delta_0(s) \) is bounded:
The MΔ Structure

- The diagram:

    - is a special case of the MΔ structure

    \[
    \begin{align*}
    u &\rightarrow K(s) \rightarrow G(s) \rightarrow W_o(s) \rightarrow \Delta_o(s) \rightarrow + \rightarrow y \\
    - &\rightarrow K(s) \rightarrow G(s) \rightarrow + \rightarrow y
    \end{align*}
    \]

    - So what?
      Later, we will see that this structure is useful for calculating robust stability.
• Step 1: flip the diagram around so that the signals flow from left to right through the perturbation.
Step 2: isolate the perturbation. Label the signals $u_\Delta$ and $y_\Delta$ and the system $M$. 

Diagram:

- $\Delta_0(s)$
- $W_0(s)$
- $G(s)$
- $K(s)$
Step 3: Compute $M$ (see Section 3.2 in your text for help).

From the diagram,

$$y = G u + u_\Delta$$

$$u = -K y$$

$$y_\Delta = W_0 G u$$
Eliminate $y$ by substituting

$$u = -K (G u + u_\Delta)$$

then

$$u = -(I+KG)^{-1}K u_\Delta$$

Then combine result with $y_\Delta = W_0 G u$, to get

$$y_\Delta = -W_0 G (I+KG)^{-1}K u_\Delta$$

in other words,

$$M = -W_0 G (I+KG)^{-1}K$$
$$= -W_0 GK (I+GK)^{-1}$$  \(\text{from Chapter 3}\)
Step 4: Finished!

- Remark: if $\Delta_0$ is completely unstructured, then the negative sign in $M$ is not relevant since both $+\Delta_0$ and $-\Delta_0$ are permitted.
Section 8.6.1 of the textbook lists $M$ and $\Delta$ for six common configurations of uncertainty.

The same procedure applies for configuring an $M\Delta$ for multiple perturbations in a system $\{\Delta_1, \Delta_2, \text{etc}\}$.

Your assignment will give you an opportunity to practice!
The MΔ Structure

- If there are multiple sources of model uncertainty \{Δ₁, Δ₂, etc\} in a system,

![Diagram](image)

then it is often convenient to construct an MΔ structure such that Δ is block diagonal.

\[
Δ = \begin{bmatrix}
Δ₁ & & \\
& Δ₂ & \\
& & \ddots
\end{bmatrix}
\]

Why?
It typically makes the μ-analysis for robust stability easier.
Robust Stability
• So far, we have discussed:
  – matrix perturbations for model uncertainty.
  – a structure for separating the nominal system from the perturbations.

• Next, we will see how this structure is used to give us a robust stability condition.
Nominal Stability

• Remember* that for a multivariable feedback system nominal stability (NS) was defined for \( L(s) \) with \( P_{ol} \) open-loop unstable poles:

\[
\det(I - L(j\omega)) \text{ makes } P_{ol} \text{ anticlockwise encirclements of the } \{0,0\} \text{ point.}
\]

• The closed-loop is stable iff:
  • \( \det(I - L(j\omega)) \neq 0 \)

*Section 4.9.2 in textbook.
Robust Stability

- If the nominal system $M(s)$ and the perturbation $\Delta(s)$ are both stable,

then the closed-loop system with $L = M\Delta$ is robustly stable (RS) if and only if:

- $\det(I - M(j\omega) \Delta(j\omega))$ does not encircle $\{0,0\}$ for any allowed perturbation $\Delta(s)$.

- $\det(I - M(j\omega) \Delta(j\omega)) \neq 0$ for all $\omega$, and all $\Delta(s)$
Robust stability means that none of the curves generated from the perturbations $\Delta(s)$ encircle or touch the origin $\{0,0\}$.
Two Bad Examples

Bad!
System is open-loop stable, no encirclements of \{(0,0)\} are permitted.

Bad!
Never touch the origin!
Robust Stability Comments

- How we calculate RS depends on the type of model uncertainty that we have.
  - Multiple model uncertainty would require to check the NS of each potential model.
  - With complex matrix perturbations $||\Delta(s)||_\infty \leq 1$, we need only check the second condition.

Why?
Because if all stable $\Delta(s)$ with $||\Delta(s)||_\infty \leq 1$ are allowable and there exists an allowable $\Delta(s)$ such that $\text{det}(I - M(j\omega) \Delta(j\omega))$ encircles $\{0,0\}$, then there also exists an allowable $\Delta(s)$ such that $\text{det}(I - M(j\omega) \Delta(j\omega)) = 0$. 
Assume the nominal system $M(s)$ is stable (NS), and perturbations $\Delta(s)$ are stable. Then the $M\Delta$-system is stable for all perturbations with $\|\Delta(s)\|_\infty \leq 1$, if and only if,

$$\bar{\sigma}(M(j\omega)) < 1, \quad \text{for all } \omega$$

Proof: Satisfying this condition implies that $\det(I - M(j\omega) \Delta(j\omega)) \neq 0$ for any $\omega$ and $\|\Delta(s)\|_\infty \leq 1$. (See textbook Section 8.6.)
Recall the multiplicative output uncertainty,

\[
\Delta_y \triangleq \Delta_O \quad \text{with} \quad M = -W_O GK (I + GK)^{-1}
\]

Then assuming that \( M \) is stable, we have the RS condition:

\[
\bar{\sigma}(W_O \cdot GK[I + GK]^{-1}) < 1 \quad \text{for all } \omega
\]
Example

- Notice that this condition is is terms of the complementary sensitivity function

\[ T = GK[I+GK]^{-1} \]

then

\[ \bar{\sigma}(W_o \cdot T) < 1 \quad \text{for all } \omega \]

- and bears a strong resemblance to the RS condition for multiplicative uncertainty in the SISO case.
Conclusions

- We generalized the concept of stable, bounded SISO transfer function perturbations to stable, unstructured transfer matrices bounded with $\|\Delta(s)\|_\infty \leq 1$.

- Discussed its use in common configurations of nominal models with perturbations.
Conclusions

• Presented the MΔ structure to generalize the concept of model uncertainty.

• We then used the MΔ structure to derive a robust stability condition for unstructured matrix perturbations. (I.e. it is not a general definition of RS.)

• This condition simplifies to the SISO case (last week’s lecture) when we consider transfer matrices of size 1-by-1.