Model Uncertainty and Robust Stability for Multivariable Systems

ELEC 571L – Robust Multivariable Control

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Outline

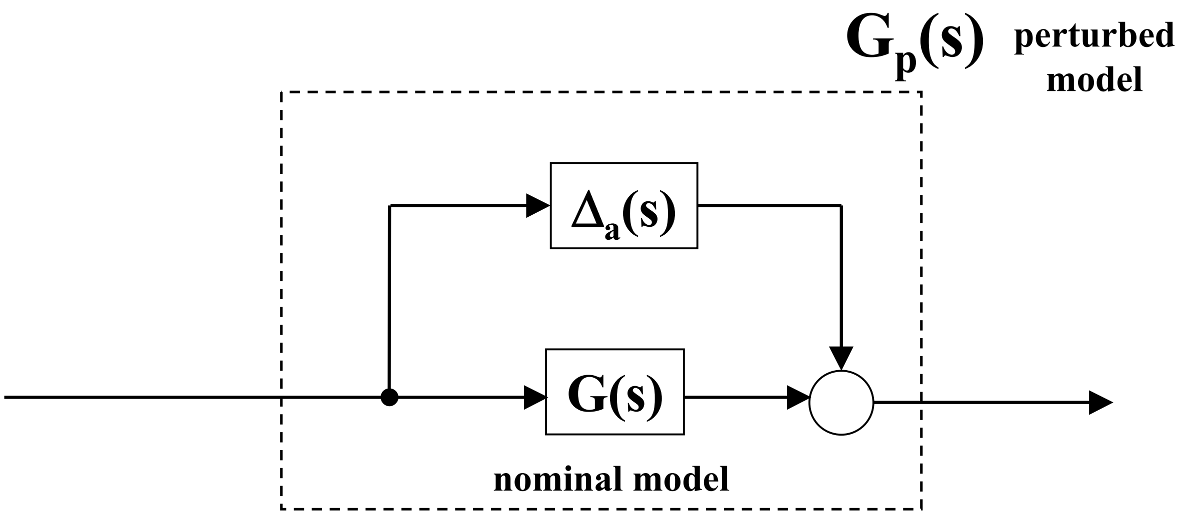
- **Representing model uncertainty.**
- **Common configurations of unstructured uncertainty.**
- **Generalizing uncertainty: the MΔ-structure.**
- **Robust stability.**

Representing
Model Uncertainty

Representing Model Uncertainty

- One common convention is to use complex transfer matrix perturbations.

- For example, additive uncertainty is given by:



- where the stable transfer matrix $\Delta_a(s)$ has the same dimension as the nominal model G(s)

Unstructured Transfer Matrices

- The assumption of completely unstructured uncertainty is represented by a bound on the maximum singular value:

$$\overline{\sigma}(\Delta(j\omega)) \leq 1 \qquad \text{for all } \omega$$

- We are assuming even less knowledge about this perturbation than we did for the SISO case.

- This transfer matrix may have any internal structure at all.

Unstructured Transfer Matrices

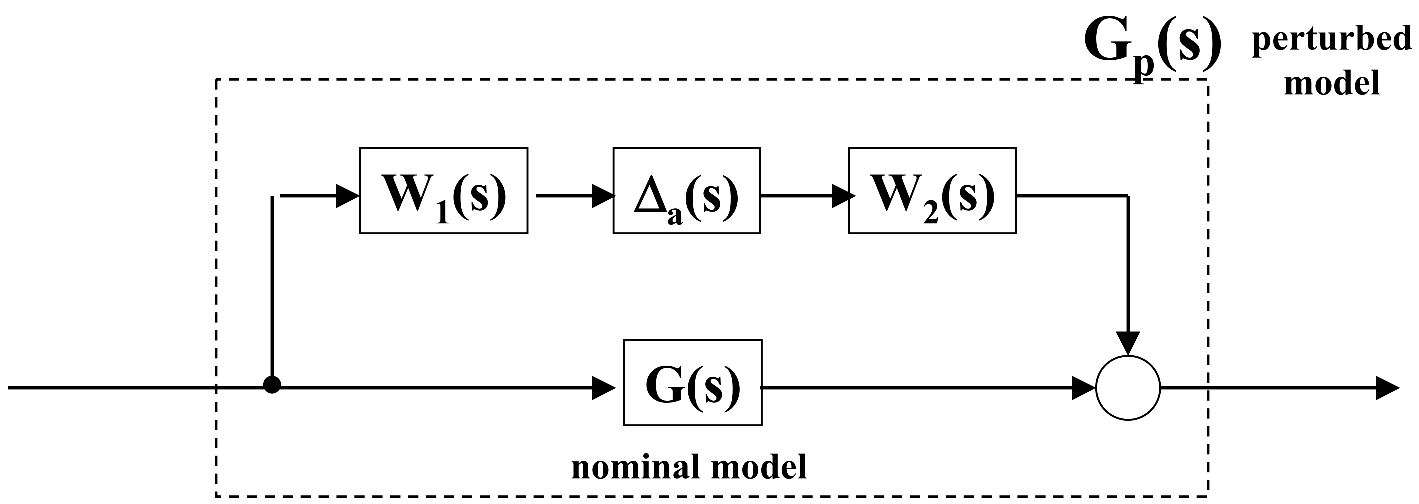
- The assumption of stability and the bound

$$\overline{\sigma}(\Delta(j\omega)) \leq 1$$

- permits *any* of the following for $\Delta(s)$:
  - Zero matrices,
  - Complex matrices,
  - Real matrices,
  - Diagonal matrices,
  - Ill-conditioned matrices,
  - Well-conditioned matrices,
  - Full matrices,
  - Sparse matrices,
  - etc.

Shaping Uncertainty

- Similar to the SISO case, the model uncertainty will often be shaped with known transfer matrices,



- Gain of W_2 and W_1 should be large for uncertain frequencies and directions of the model.

Notation

- **The bound on the largest singular value**

$$\bar{\sigma}(\Delta(j\omega)) \leq 1 \quad \text{for all } \omega$$

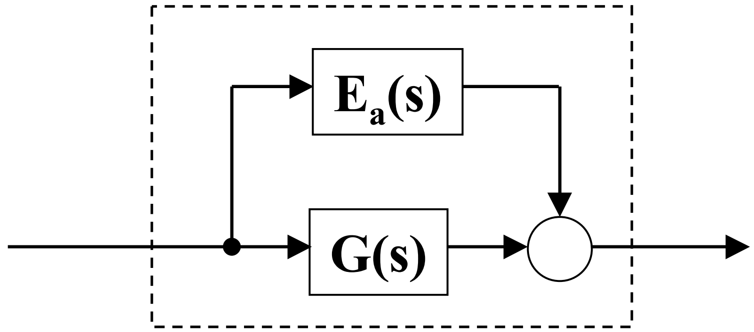
is equivalent to bounding the *H_∞* norm

$$\|\Delta(s)\|_\infty \leq 1$$

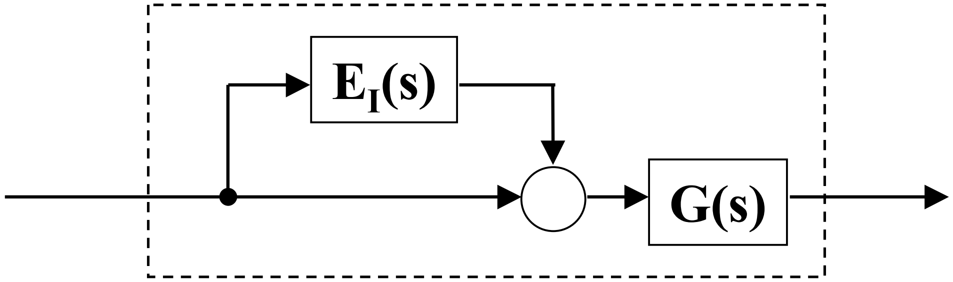
and is commonly used as notation.

Common Configurations of Model Uncertainty

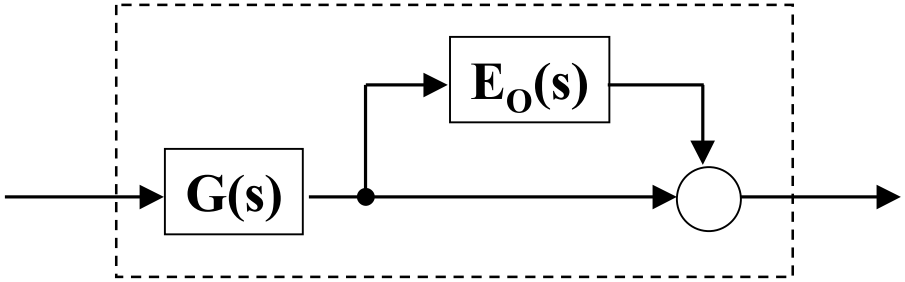
Three examples



- Additive uncertainty



- Multiplicative input uncertainty



- Multiplicative output uncertainty

Importance of Configuration

- In MIMO systems, it is more difficult to move uncertainty around than in SISO systems.

- Consider for example, the three previous cases:

$$G_p = G + E_A \qquad \text{(additive)}$$

$$G_p = G\,(I + E_I) \qquad \text{(multiplicative input)}$$

$$G_p = (I + E_O)\,G \qquad \text{(multiplicative output)}$$

The Importance of Configuration

- Suppose that we have $G_p = G\ (I + E_I)$, then the size of the uncertainty is $\| E_I \|_\infty$

- Now, suppose we wish to look at uncertainty in terms of E_O.

- The two multiplicative configurations may be made equivalent by writing

$$E_O\, G = G\, E_I$$

- Then (ignoring issues of invertibility),

$$E_O = G\, E_I\, G^{-1}$$

The Importance of Configuration

- **Then taking the maximum singular value of both sides,**

$$\overline{\sigma}(E_O) = \overline{\sigma}(G \cdot E_I \cdot G^{-1})$$

- **If we assume that the perturbations are unstructured, then we can "pull-out"**

$$\overline{\sigma}(E_O) = \overline{\sigma}(E_I) \cdot \overline{\sigma}(G) \cdot \overline{\sigma}(G^{-1})$$

$$= \overline{\sigma}(E_I) \cdot \frac{\overline{\sigma}(G)}{\underline{\sigma}(G)}$$

The Importance of Configuration

- **If the matrix G is ill-conditioned, then $\underline{\sigma}$(G) is small,**

$$\bar{\sigma}(E_O) = \bar{\sigma}(E_I) \cdot \frac{\bar{\sigma}(G)}{\underline{\sigma}(G)}$$

and therefore $\bar{\sigma}(E_O)$ **must be large!**

- **A similar effect would have happened if we had tried to represent a E_O perturbation in terms of E_I.**

On the Other Hand...

- **We can represent *both* input and output multiplicative uncertainty easily in terms of additive uncertainty.**

- **First, we equate the perturbations,**

$$E_A \quad = \quad E_O \, G \quad = \quad G \, E_I$$

On the Other Hand...

- Then, for unstructured matrices $\{E_A, E_O, E_I\}$, we find:

$$\bar{\sigma}(E_A) = \bar{\sigma}(E_O)\bar{\sigma}(G) = \bar{\sigma}(G)\bar{\sigma}(E_I)$$

and since we only deal with the maximum singular value (i.e. not the minimum), then we are insensitive to ill-conditioned matrices G.

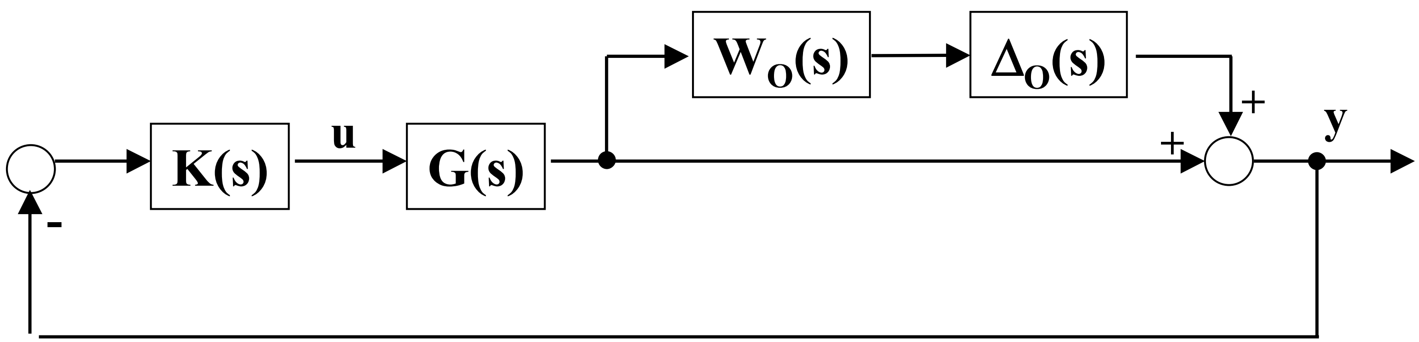
- Those of you doing the CD control project should take note of this result!

Generalizing Uncertainty:

the MΔ-Structure

A Familiar Structure

- Consider a typical MIMO feedback loop with multiplicative output uncertainty:

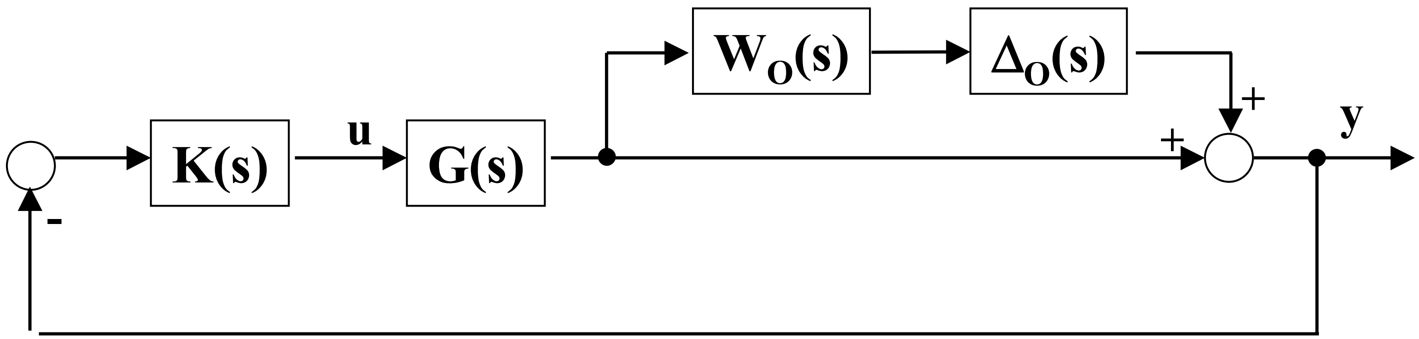


- the unstructured, stable transfer matrix $\Delta_O(s)$ is bounded:

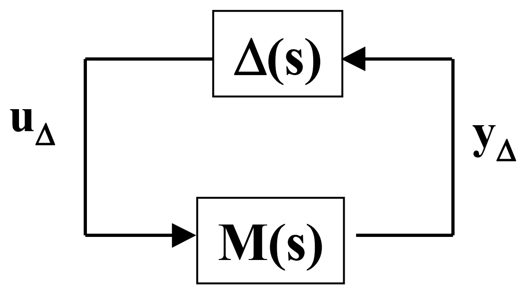
$$||\Delta_O(s)||_\infty \leq 1$$

The MΔ Structure

- The diagram:



is a special case of the MΔ structure

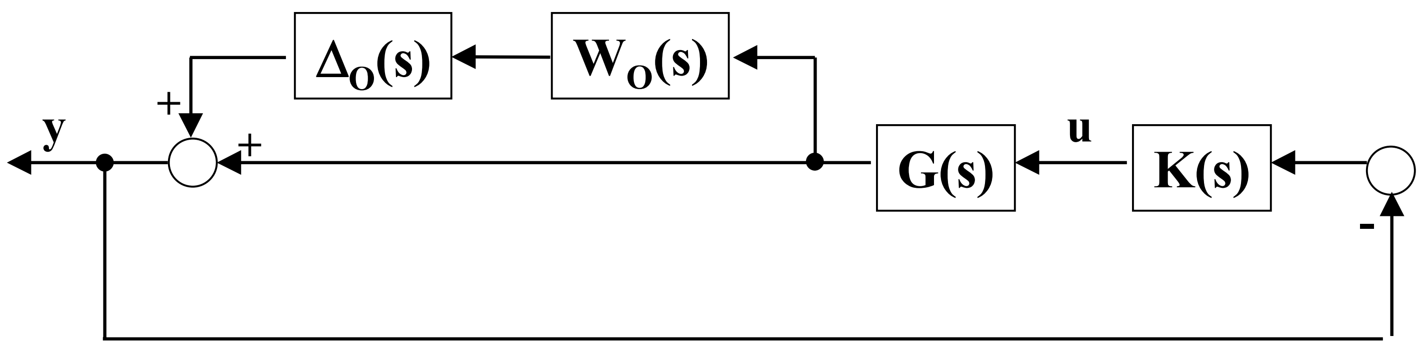


So what?

Later, we will see that this structure is useful for calculating robust stability.

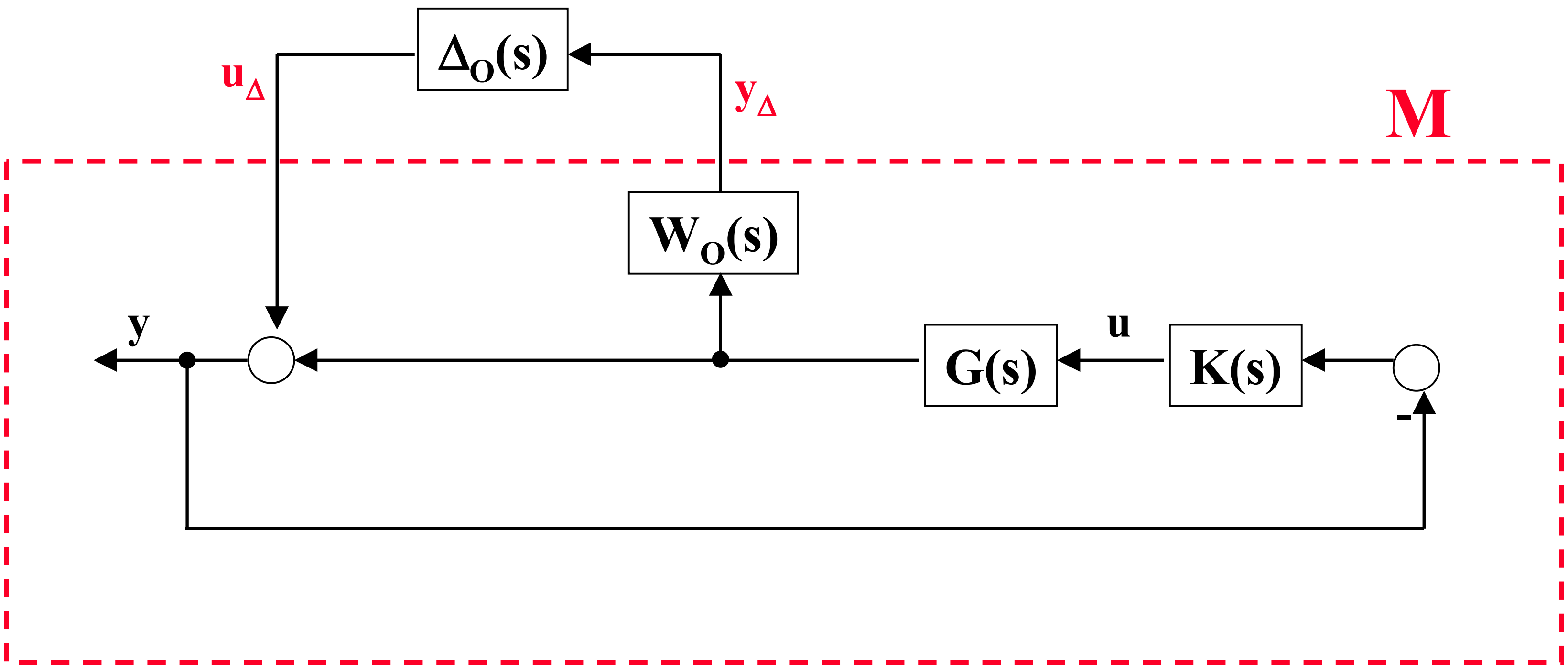
One Approach to Reconfiguring

- **Step 1: flip the diagram around so that the signals flow from left to right through the perturbation.**



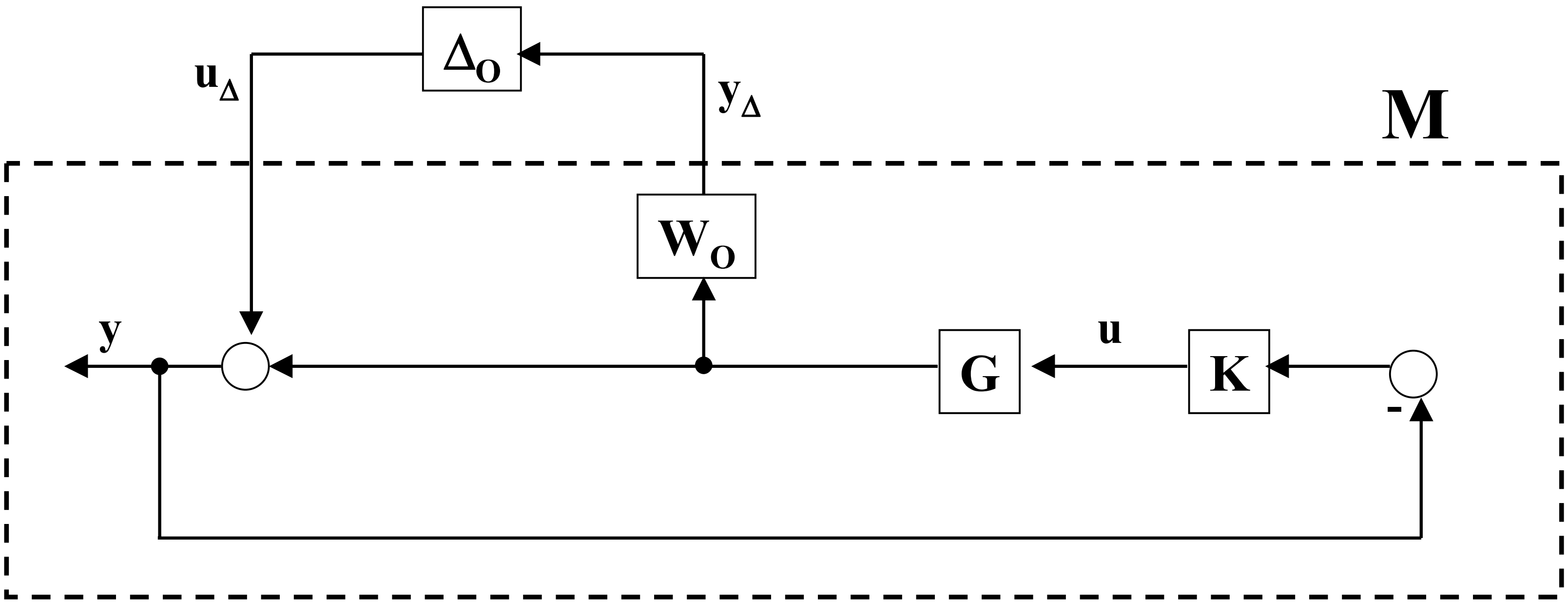
One Approach to Reconfiguring

- Step 2: isolate the perturbation. Label the signals u_Δ and y_Δ and the system M.



One Approach to Reconfiguring

- **Step 3: Compute M** (see Section 3.2 in your text for help).



From the diagram,

$$y = G\,u + u_\Delta$$

$$u = -K\,y$$

$$y_\Delta = W_O\,G\,u$$

One Approach to Reconfiguring

Eliminate y by substituting

$$u = -K\,(G\,u + u_\Delta)$$

then

$$u = -(I+KG)^{-1}K\,u_\Delta$$

Then combine result with $y_\Delta = W_O\,G\,u$, to get

$$y_\Delta = -W_O\,G\,(I+KG)^{-1}\,K\,u_\Delta$$

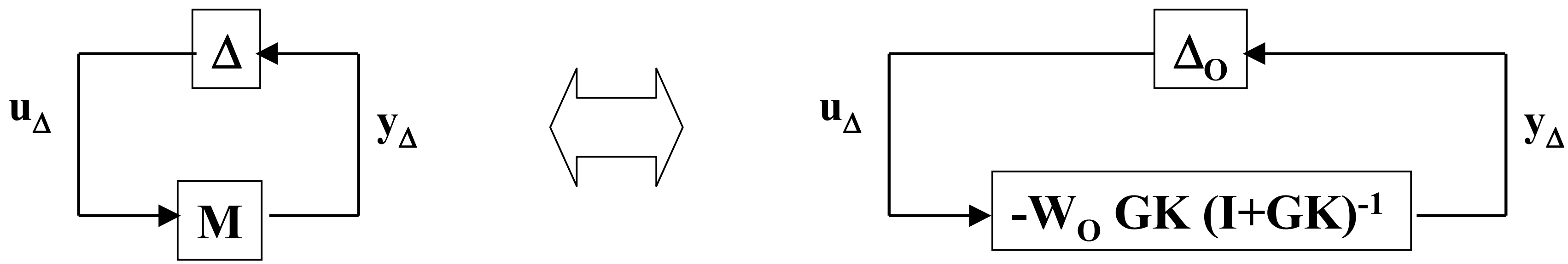
in other words,

$$\begin{aligned} M &= -W_O\,G\,(I+KG)^{-1}\,K \\ &= -W_O\,GK\,(I+GK)^{-1} \end{aligned}$$

(from Chapter 3)

One Approach to Reconfiguring

- **Step 4: Finished!**



- **Remark: if Δ_O is completely unstructured, then the negative sign in M is not relevant since both $+\Delta_O$ and $-\Delta_O$ are permitted.**

MΔ-Structure - Comments

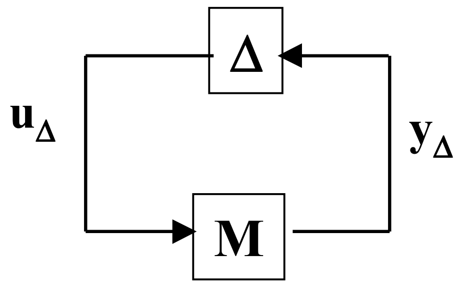
- Section 8.6.1 of the textbook lists M and Δ for six common configurations of uncertainty.

- The same procedure applies for configuring an MΔ for multiple perturbations in a system {Δ_1, Δ_2, etc}.

- Your assignment will give you an opportunity to practice!

The MΔ Structure

- If there are multiple sources of model uncertainty $\{\Delta_1, \Delta_2, \text{etc}\}$ in a system,



then it is often convenient to construct an MΔ structure such that Δ is block diagonal.

$$\Delta = \begin{bmatrix} \Delta_1 & & \\ & \Delta_2 & \\ & & \ddots \end{bmatrix}$$

Why?

It typically makes the μ-analysis for robust stability easier.

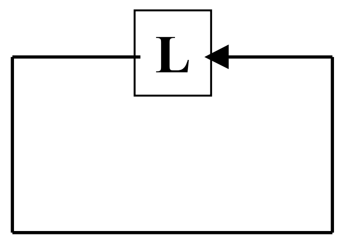
Robust Stability

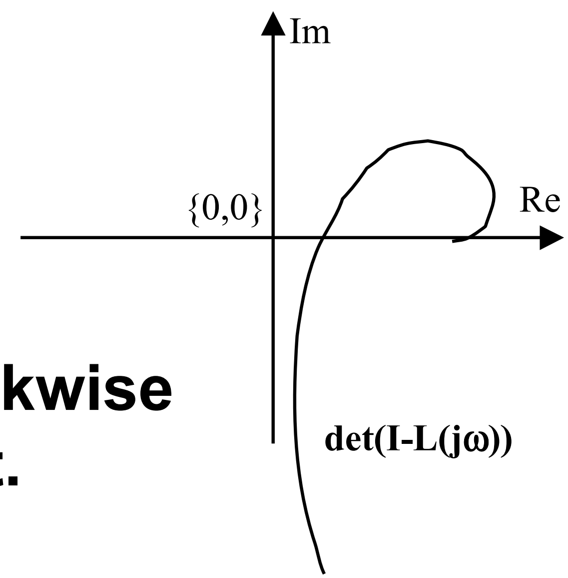
Robust Stability

- So far, we have discussed:
  - matrix perturbations for model uncertainty.
  - a structure for separating the nominal system from the perturbations.
- Next, we will see how this structure is used to give us a robust stability condition.

Nominal Stability

- Remember* that for a multivariable feedback system nominal stability (NS) was defined for L(s) with P_{ol} open-loop unstable poles:



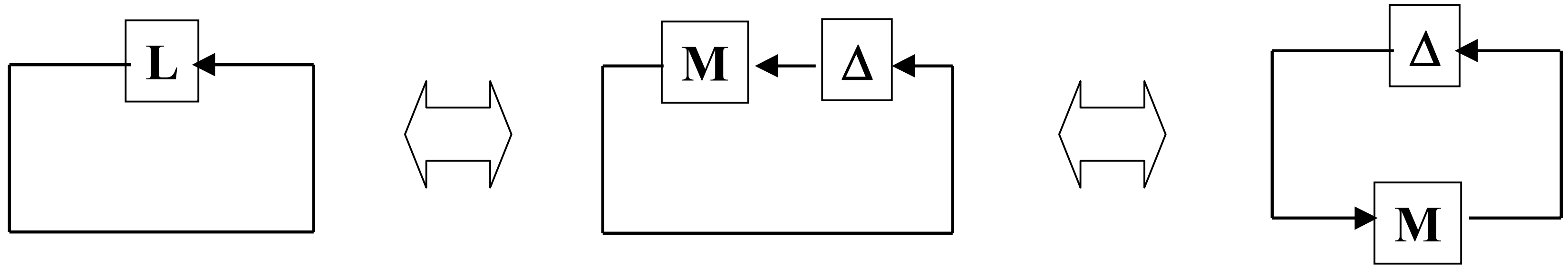


- The closed-loop is stable iff:
  - $\det(I - L(j\omega))$ makes P_{ol} anticlockwise encirclements of the {0,0} point.
  - $\det(I - L(j\omega)) \neq 0$

*Section 4.9.2 in textbook.

Robust Stability

- If the nominal system M(s) and the perturbation Δ(s) are both stable,



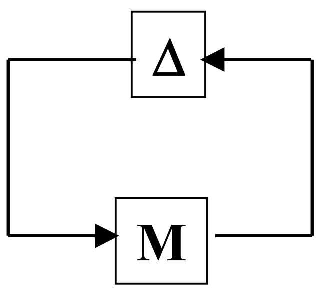
then the closed-loop system with L = MΔ is robustly stable (RS) if and only if:

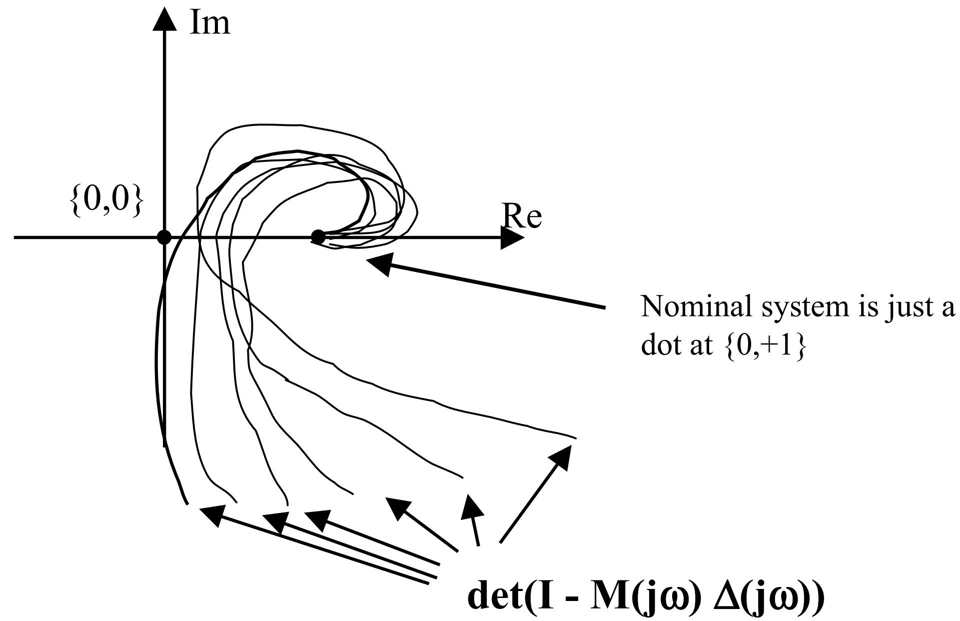
- det(I - M(jω) Δ(jω)) does not encircle {0,0} for any allowed perturbation Δ(s).

- det(I - M(jω) Δ(jω)) ≠ 0 for all ω,

  and all Δ(s)

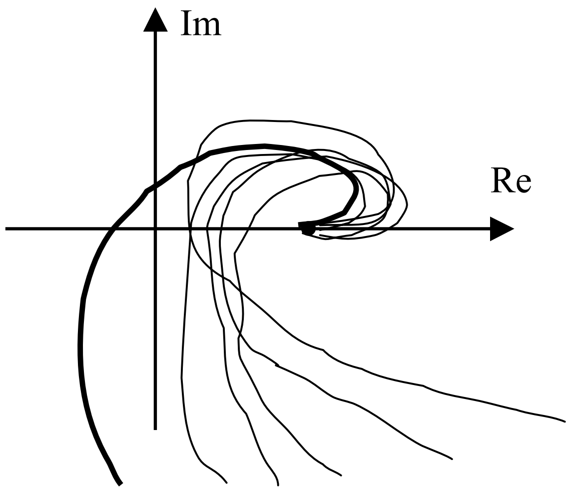
Robust Stability Graphical Interpretation



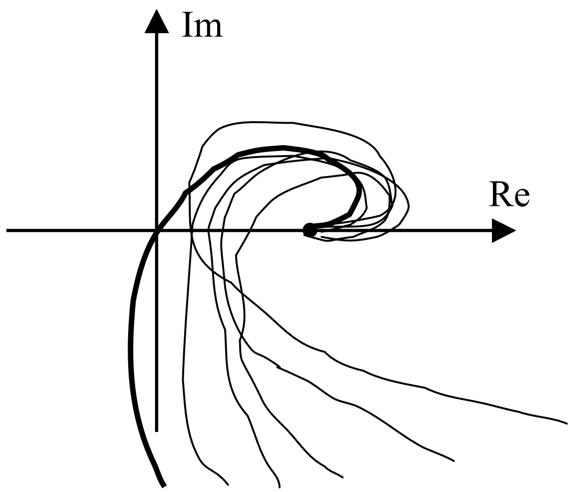


Robust stability means that none of the curves generated from the perturbations $\Delta(s)$ encircle *or* touch the origin {0,0}.

Two Bad Examples



Bad!
System is open-loop stable, no encirclements of {0,0} are permitted.



Bad!
Never touch the origin!

Robust Stability Comments

- How we calculate RS depends on the type of model uncertainty that we have.
  - Multiple model uncertainty would require to check the NS of each potential model.
  - With complex matrix perturbations $||\Delta(s)||_\infty \leq 1$, we need only check the second condition.

Why?
Because *if* all stable $\Delta(s)$ with $\|\|\Delta(s)\|\|_\infty \leq 1$ are allowable and there exists an allowable $\Delta(s)$ such that $\det(I - M(j\omega)\,\Delta(j\omega))$ encircles {0,0}, *then* there also exists an allowable $\Delta(s)$ such that $\det(I - M(j\omega)\,\Delta(j\omega)) = 0$.

Robust Stability for Unstructured Uncertainty

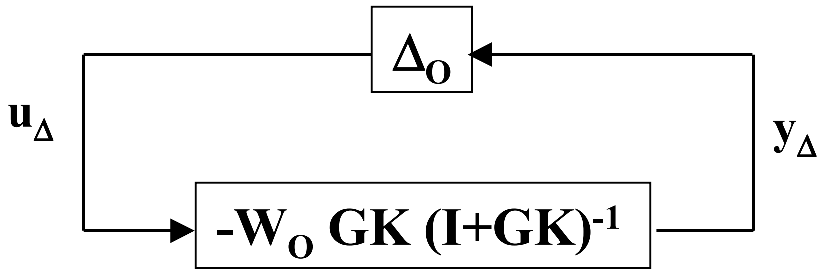
Assume the nominal system M(s) is stable (NS), and perturbations $\Delta(s)$ are stable. Then the $M\Delta$-system is stable for all perturbations with $||\Delta(s)||_\infty \le 1$, if and only if,

$$\bar{\sigma}(M(j\omega)) < 1, \qquad \text{for all } \omega$$

Proof : Satisfying this condition implies that det(I - M($j\omega$) Δ($j\omega$)) $\neq$ 0 for any ω and $||\Delta(s)||_\infty \le 1$. (See textbook Section 8.6.)

Example

- **Recall the the multiplicative output uncertainty,**



with $\quad M = -W_O\ GK\ (I+GK)^{-1}$

- **Then assuming that M is stable, we have the RS condition:**

$$\overline{\sigma}(W_O \cdot GK[I+GK]^{-1}) < 1 \quad \text{for all}\ \omega$$

Example

- **Notice that this condition is is terms of the complementary sensitivity function**

$$T = GK[I+GK]^{-1}$$

then

$$\bar{\sigma}(W_O \cdot T) < 1 \quad \text{for all}\, \omega$$

- **and bears a strong resemblance to the RS condition for multiplicative uncertainty in the SISO case.**

Conclusions

- **We generalized the concept of stable, bounded SISO transfer function perturbations to stable, unstructured transfer matrices bounded with $||\Delta(s)||_\infty \leq 1$.**

- **Discussed its use in common configurations of nominal models with perturbations.**

Conclusions

- **Presented the MΔ structure to generalize the concept of model uncertainty.**
- **We then used the MΔ structure to derive a robust stability condition for *unstructured matrix perturbations*.** **(I.e. it is not a <u>general</u> definition of RS.)**
- **This condition simplifies to the SISO case (last week's lecture) when we consider transfer matrices of size 1-by-1.**