Structured Uncertainty and Robust Performance

ELEC 571L – Robust Multivariable Control

prepared by: Greg Stewart

Honeywell

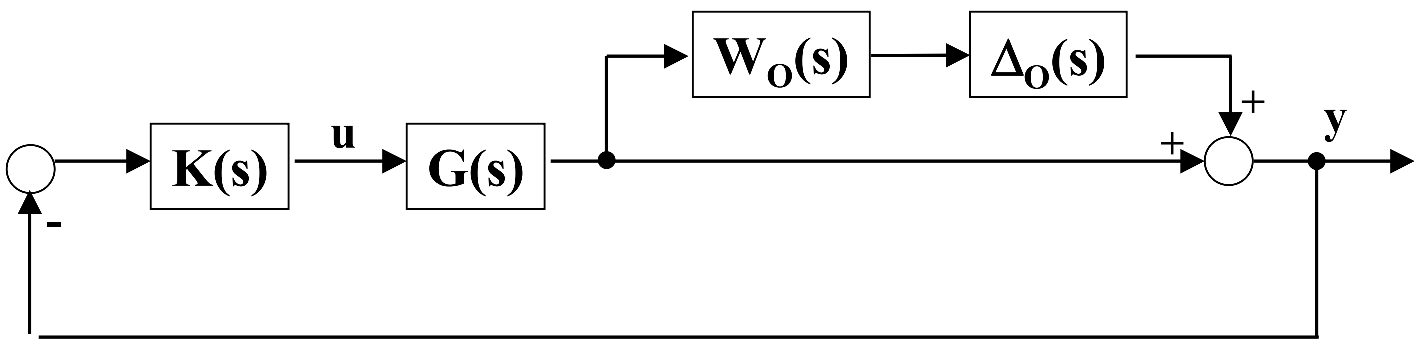
Outline

- Structured uncertainty: motivating example.
- Structured singular value, μ.
- Robust stability for structured uncertainty.
- Application of μ: robust performance.

Motivating Example

From last week...

- Consider the usual MIMO feedback loop with multiplicative output uncertainty:

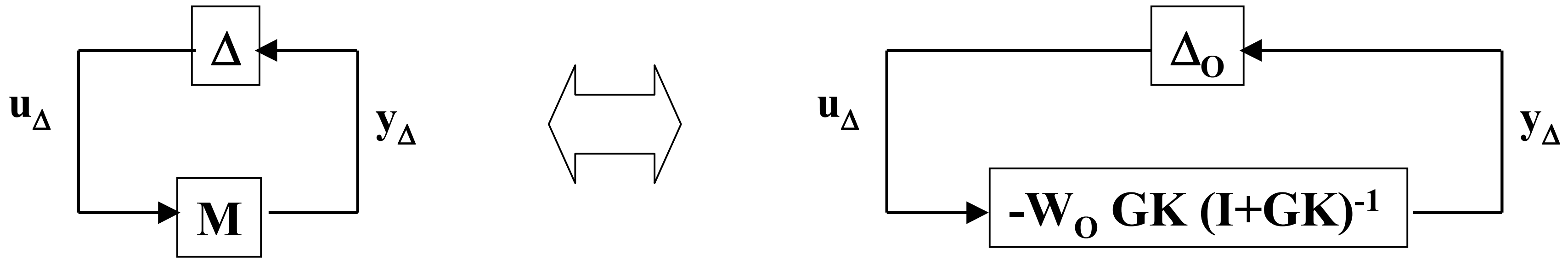


- the unstructured, stable transfer matrix $\Delta_O(s)$ is bounded:

$$\|\Delta_O(s)\|_\infty \leq 1$$

Robust Stability for Unstructured Uncertainty

- **After massaging the block diagrams, we derived the MΔ structure:**

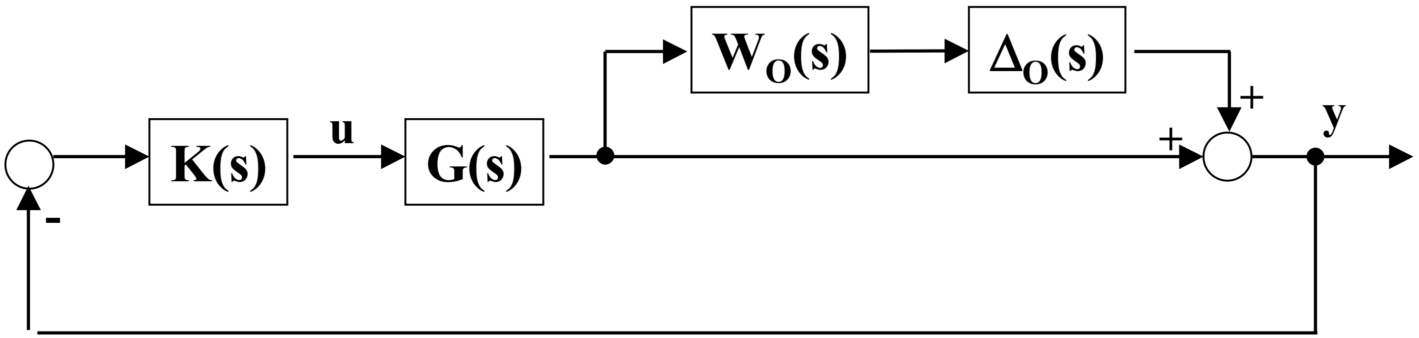


- **and then said that "if we allow *all* Δ_O such that $||\Delta_O||_\infty \leq 1$, then robust stability (RS) is given by NS and**

$$\overline{\sigma}(W_O \cdot GK[I+GK]^{-1}) < 1 \quad \text{for all } \omega$$

Example

- **Assume:**
  - we have a 2-by-2 system (two actuators and two sensors).
  - we are very confident in our knowledge of our process.
  - However, the sensors were supplied by a very shady vendor (i.e. not Honeywell), and we only trust the readings by ±30%.



Example: model the uncertainty

- **therefore we decide to go with:**

$$W_O = \begin{bmatrix} 0.3 & 0 \\ 0 & 0.3 \end{bmatrix}, \qquad \Delta_O = \begin{bmatrix} \Delta_1 & 0 \\ 0 & \Delta_2 \end{bmatrix}$$

where each diagonal element is SISO and bounded,

$$|\Delta_1(j\omega)| \leq 1, \qquad |\Delta_2(j\omega)| \leq 1$$

giving the overall matrix $||\Delta_O||_\infty \leq 1$.

(see textbook Appendix for the proof!)

Analysis

- Now suppose that at some frequency ω, we find that M has the numerical values:

$$M = W_O \cdot GK[I + GK]^{-1}$$

$$= \begin{bmatrix} 0.1860 & 0.3719 \\ 0.5579 & 0.7438 \end{bmatrix}$$

Analysis

- **We then compute the maximum singular value and find**

$$\bar{\sigma}\left(M(j\omega)\right) = \bar{\sigma}\left(\begin{bmatrix} 0.1860 & 0.3719 \\ 0.5579 & 0.7438 \end{bmatrix}\right)$$

$$= 1.0162$$

- **this is larger than 1 (and therefore violates last week's condition for RS).**
- **Does this mean we are not robustly stable?**

Analysis

- **The answer is NO, we are not necessarily violating robust stability.**

- **What it DOES say, is that there exists SOME stable matrix $\Delta_O(s)$ with $||\Delta_O||_\infty \leq 1$, such that the M$\Delta$ structure is unstable.**

- **But this is *NOT* necessarily the same as failing RS!**

- **Maybe our uncertainty model does not permit the nasty, destabilizing perturbation $\Delta_O(s)$?**

Uncertainty Structure

- **Remember that, for this example, we are attempting to model sensor uncertainty only.**

- **We are considering <u>only</u> diagonal matrices $\Delta_O(s)$:**

$$\Delta_O = \begin{bmatrix} \Delta_1 & 0 \\ 0 & \Delta_2 \end{bmatrix}$$

- **The real question is: "Does there exist a <u>*diagonal*</u> matrix $\Delta_O(s)$ that will destabilize the MΔ structure?"**

Experimental Robust Stability

- **We will do a numerical experiment.**

- **Given the nominal system,**

$$M = \begin{bmatrix} 0.1860 & 0.3719 \\ 0.5579 & 0.7438 \end{bmatrix}$$

- **Let us check the robust stability for**
  - **(1) full matrix perturbations, Δ_{full}**
  - **(2) diagonal matrix perturbations, Δ_{diag}**

Experimental Robust Stability

- **Each of these perturbations satisfies the assumptions of Theorem 8.2.**
- **Which allows us to check RS by computing the maximum eigenvalue magnitude,**

$$\rho(M\Delta) < 1$$

- **In Matlab notation,**

```
if max(abs(eig(M*Delta))) < 1
        'Delta is not destabilizing'
end
```

Experimental Robust Stability

1. **We create full complex matrix perturbations in Matlab with the commands:**

```
Delta_full = rand(n,n) + sqrt(-1)*rand(n,n);
Delta_full = Delta_full /max(svd(Delta_full));
```

2. **Diagonal matrix perturbations with the commands:**

```
Delta_diag = diag(rand(1,n)) + sqrt(-1)*diag(rand(1,n));
Delta_diag = Delta_diag /max(svd(Delta_diag));
```

Experimental Robust Stability

- Using the program “finddelta.m”, we tested robust stability using Theorem 8.2, for 1 000 000 randomly generated full and diagonal matrices.

- Results:

Perturbation	# Destabilizing	Percentage
Δ_{full}	10 694	~1%
Δ_{diag}	0	0%

Experimental Robust Stability

- In a million examples, we found several full matrices Δ_{full} which caused the system to be unstable.
- But we did not find a single destabilizing diagonal perturbation, Δ_{diag} .
- This indicates (but does not prove) that we are safe from instability since we are only concerned with diagonal perturbations Δ_{diag} .
- Look at Example 8.9 in your text for a related example.

Experimental Robust Stability

In addition, these results also indicate (but do not prove) that the condition,

$$\overline{\sigma}(M) < 1$$

may be too conservative for use checking robust stability when we have knowledge of the *structure* of the perturbation Δ.

Structured Uncertainty

- In this case, we knew that Δ was a diagonal matrix.

- This is common as robust control typically imposes a band-diagonal structure on the Δ,

$$\Delta = \begin{bmatrix} \Delta_1 & & \\ & \Delta_2 & \\ & & \ddots \end{bmatrix}$$

with a separate Δ_i for each independent source of uncertainty.

The Structured Singular Value

Structured Singular Value: Definition

Find the smallest allowable Δ (measured in terms of σ(Δ)) which makes det(I - M Δ) = 0, then the SSV is defined as,

$$\mu(M(j\omega)) = \frac{1}{\overline{\sigma}(\Delta(j\omega))}$$

for Δ ∈ Π, where Π is a set of structured perturbations (not necessarily σ(Δ)≤1).

Robust Stability for Structured Uncertainty

Assume that the nominal system M and the perturbations Δ are stable and $||\Delta||_\infty \leq 1$. Then the MΔ-system is stable for all allowed perturbations <u>if and only if</u>,

$$\mu(M(j\omega)) < 1, \qquad \text{for all } \omega$$

for all $\Delta \in \Pi$, where Π is the set of *allowable* perturbations.

Example

- The m-file “calcmu.m” is a brute-force algorithm for computing μ.
- We randomly generated 4 000 000 perturbing matrices Δ_{full} and Δ_{diag}.
- Then check to see the smallest matrices that destabilized the $M\Delta$ system for our previous example.

Example: full perturbations

- **For Δ_{full}, the smallest destabilizing perturbation (for frequency ω) was found at 0.9872, therefore:**

$$\mu(M) \approx \frac{1}{0.9872} = 1.0130$$

- **which indicates that (at this frequency) the system is *not* robustly stable for full perturbations with $||\Delta_{full}||_\infty \leq 1$**

- **Remark: for full perturbations, we can verify this numerical value against the true μ: $\mu(M) = \overline{\sigma}(M) = 1.0162$. So it is quite close!**

Example: structured perturbations

- For Δ_{diag}, the smallest destabilizing perturbation was found at 1.0014, therefore:

$$\mu(M) \approx \frac{1}{1.0014} = 0.9986$$

- as expected, this number is smaller than for the full perturbations!
- and since it is smaller than 1, we then satisfy (at this frequency) robust stability for diagonal perturbations with $||\Delta_{diag}||_\infty \leq 1$.

Structure: Special Cases

- **In general,** $\mu(M) \le \overline{\sigma}(M)$

- **However, if the uncertainty set Π happens to contain the worst case perturbation Δ, then we get equality**

$$\mu(M) = \overline{\sigma}(M)$$

- **The most common example of this is unstructured uncertainty,**

$$\Pi = \{ \Delta : \|\Delta\|_\infty \le 1 \}$$

Structure: Special Cases

- **Multiple sources of uncertainty. Typically resulting in a block-diagonal Δ:**

$$\Pi = \{ \Delta = \text{diag}\{\Delta_1, \Delta_2, \ldots\} : \|\Delta_i\|_\infty \leq 1\}$$

- **Parametric uncertainty. Typically resulting in real Δ:**

$$\Pi = \{ \Delta \in \Re : -1 \leq \Delta \leq 1\}$$

Computation of $\mu(M)$

- **Unfortunately it is not straightforward to compute $\mu(M)$ in general.**

- **Various methods exist for computing $\mu(M)$.**

- **Methods often depend on the structure of Δ and M.**

- **It is worth reading Sections 8.8.2 and 8.8.3 to see results for some special cases which arise in practical work.**

The Structured Singular Value
and
Robust Performance

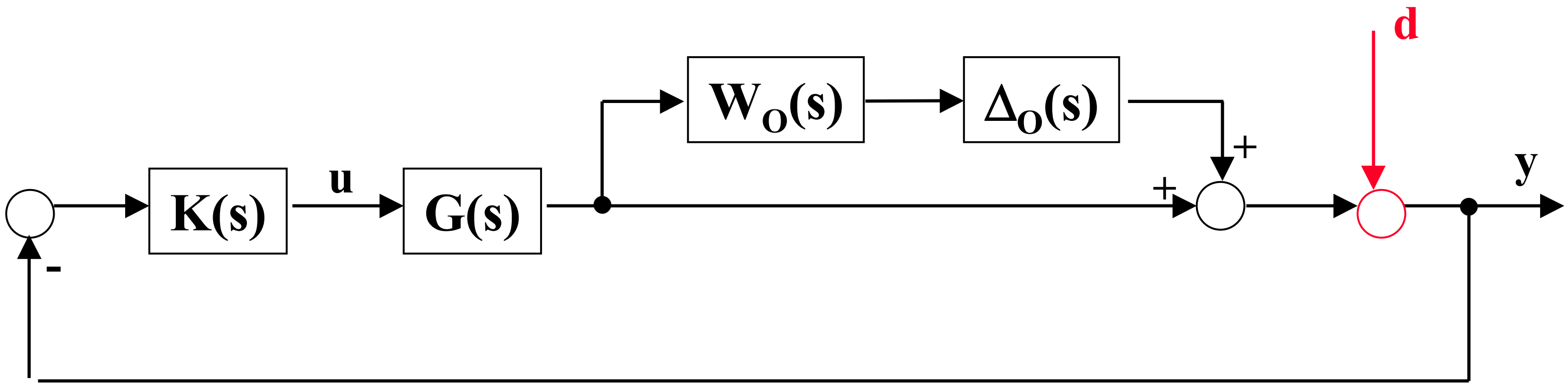
Application: Robust Performance with μ

- We just discussed using the $\mu(M)$ to test for robust stability (RS).

- If we reconfigure our blocks, we can apply the same concepts to test for robust performance (RP).

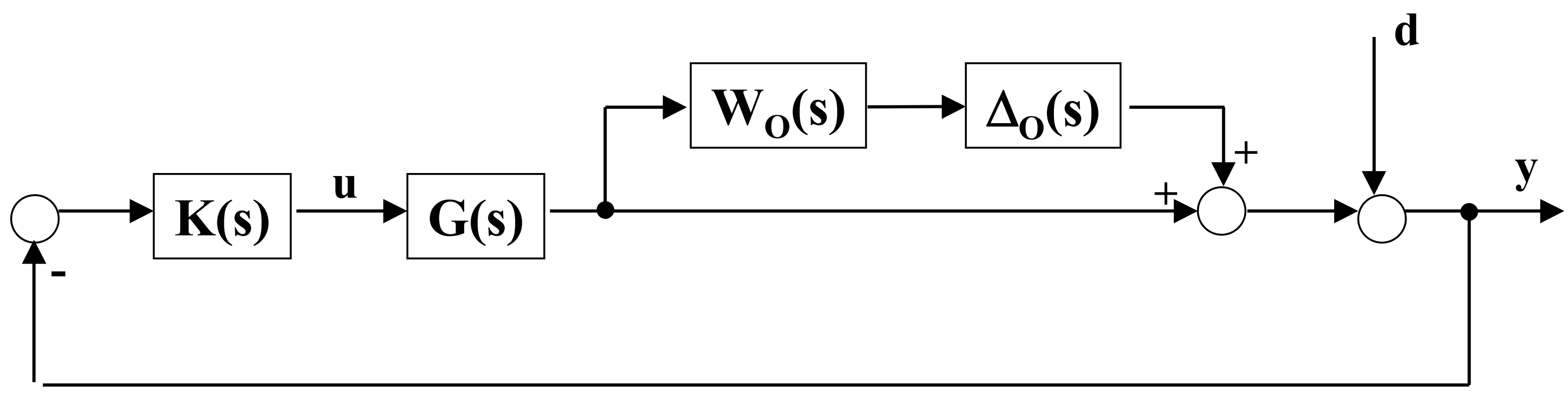
Our usual system

- Take our usual system with multiplicative output uncertainty:



- The only new feature is the additive output disturbance, d.

Define Performance



- We will use a familiar definition of performance:

$$\bar{\sigma}(W_p S) < 1$$

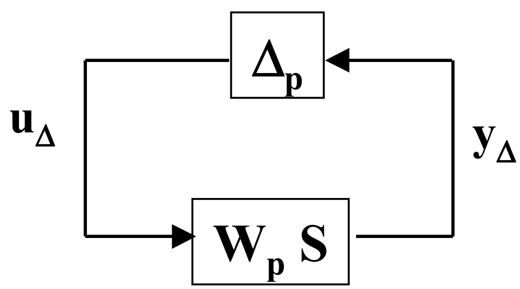
- where the sensitivity function $S = [I+G_pK]^{-1}$.
- W_p is typically a low-pass filter (often an integrator).

Re-interpret Performance

- **Note that, the performance specification:**

$$\bar{\sigma}(W_p S) < 1$$

- **Can be re-written as a RS condition using a fictitious perturbation Δ_p:**

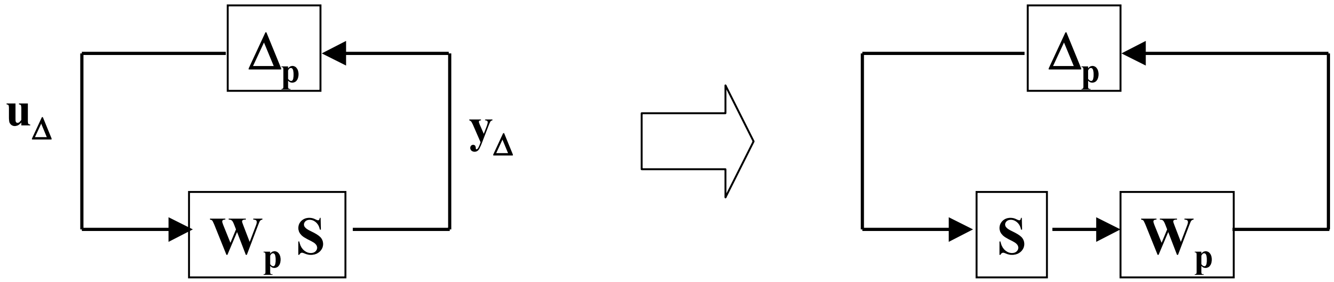


$$\overset{\textbf{RS}}{\Longleftrightarrow}$$

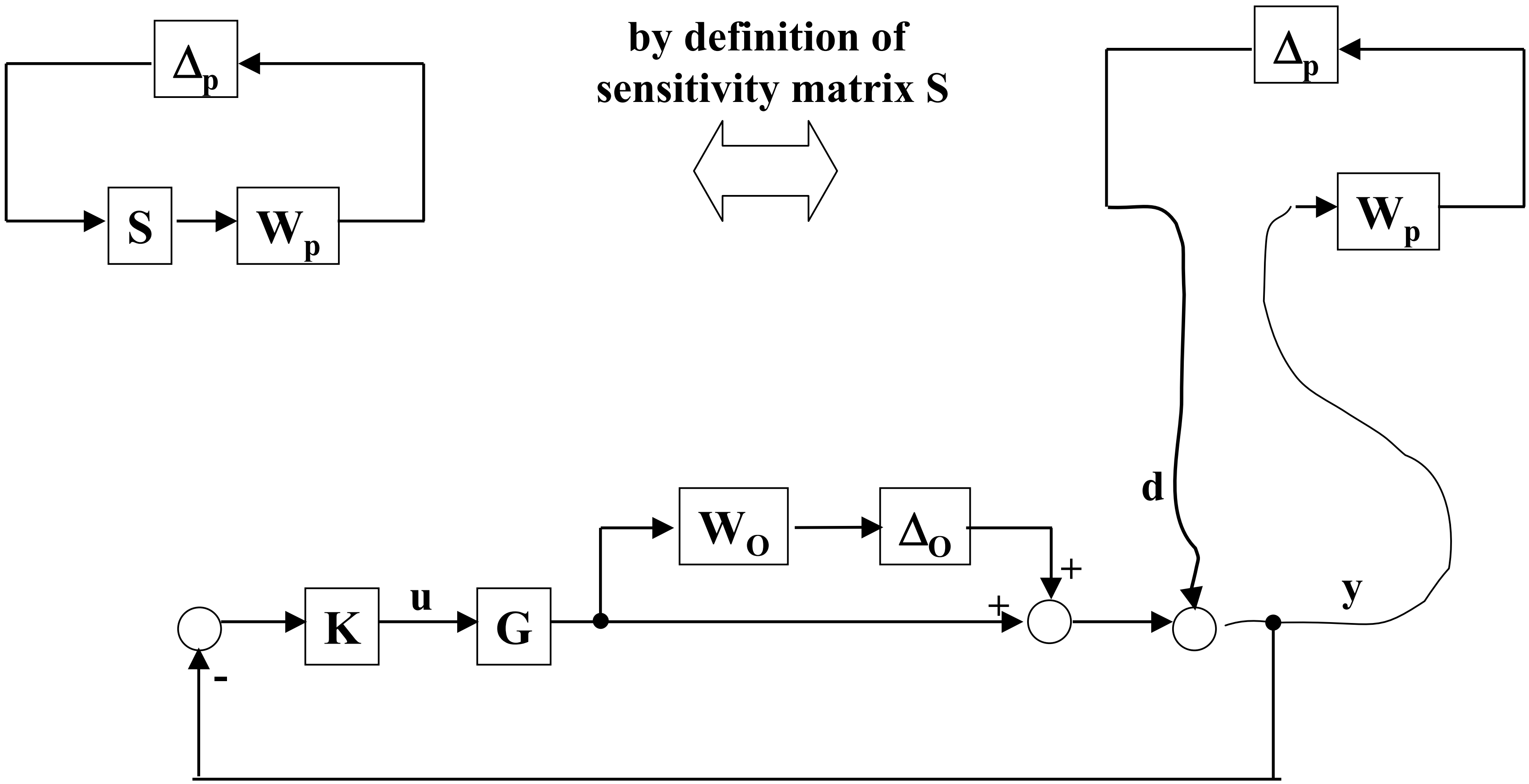
$$\bar{\sigma}(W_p S) < 1$$

with $\|\Delta_p\|_\infty \leq 1$

Reconfiguring

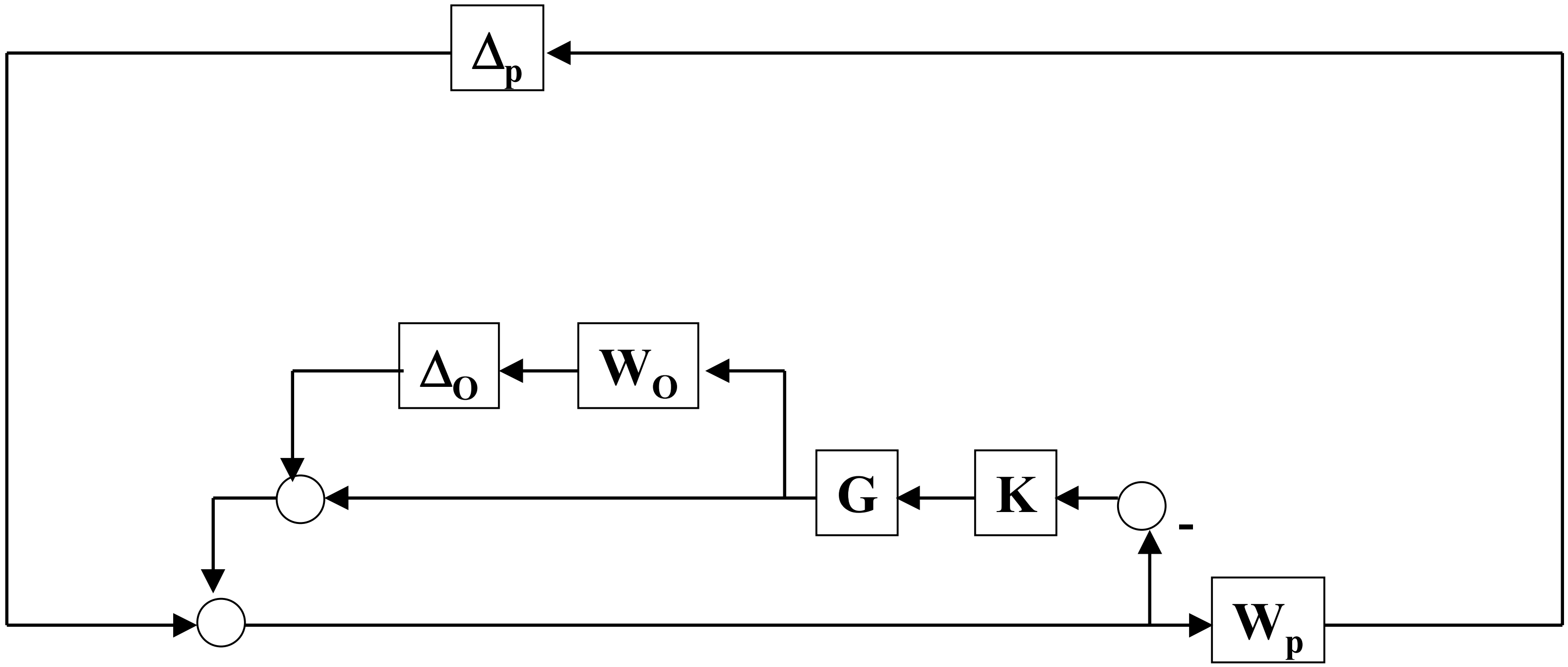


Reconfiguring

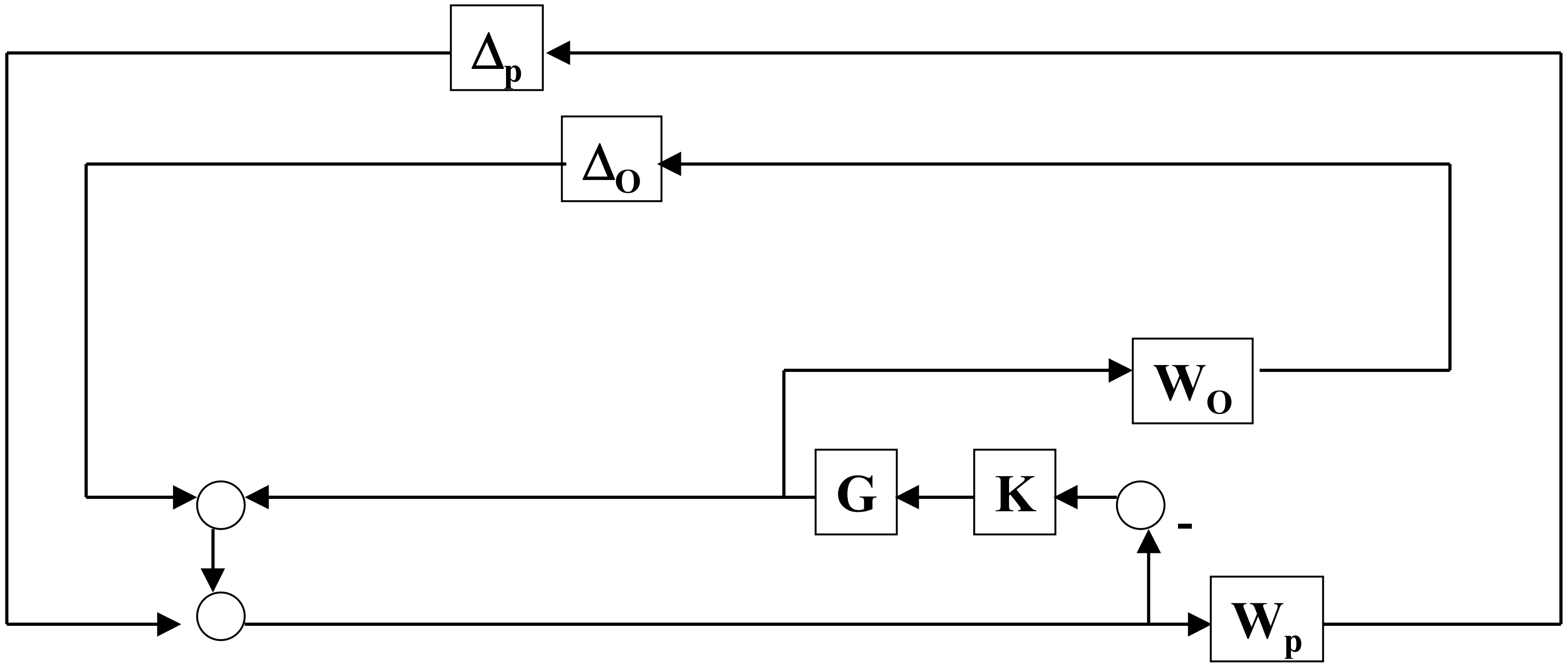


- which now has two deltas:
  - Δ_O for model uncertainty,
  - Δ_p for performance.

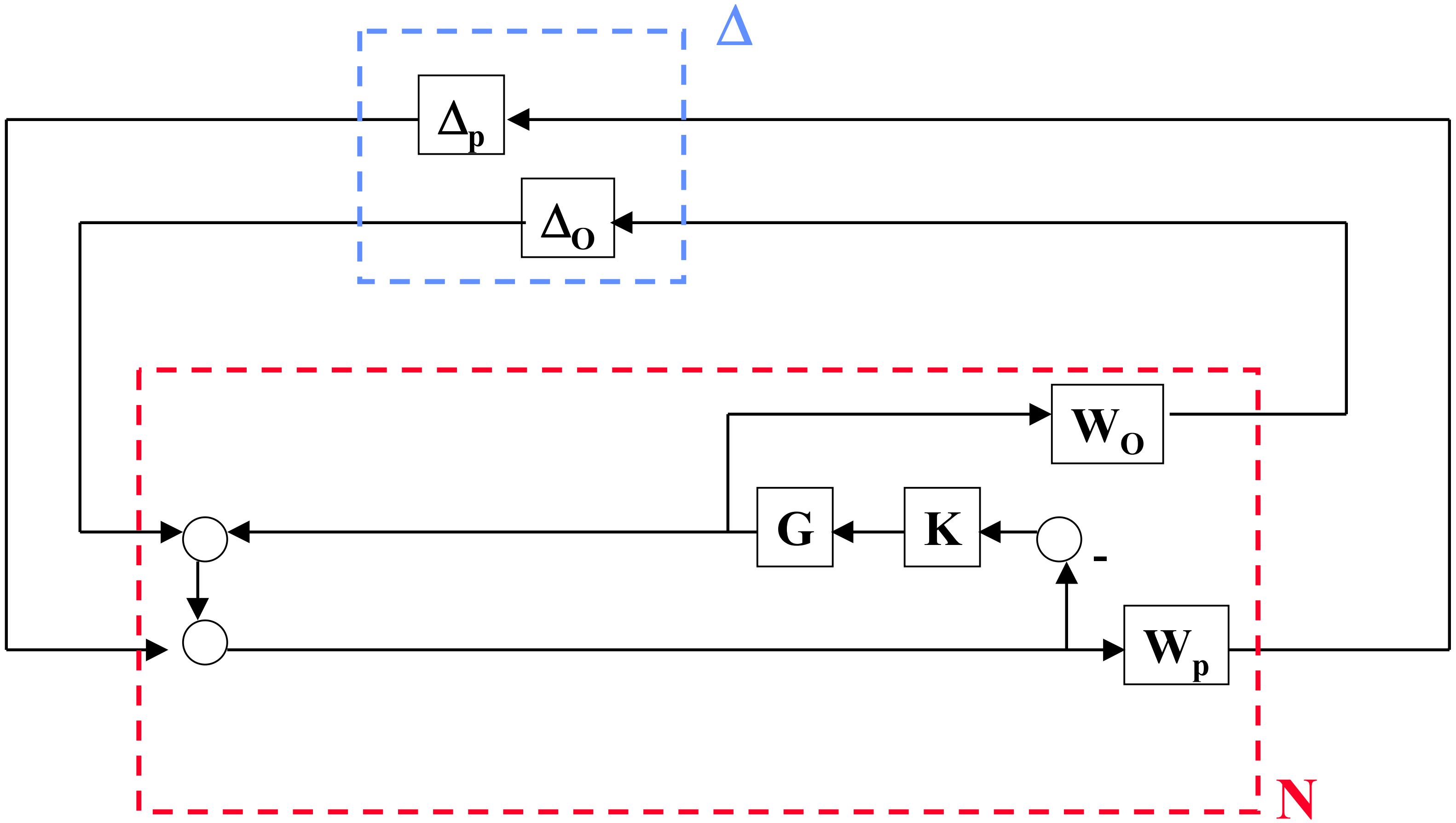
Massaging Block Diagram



Massaging Block Diagram



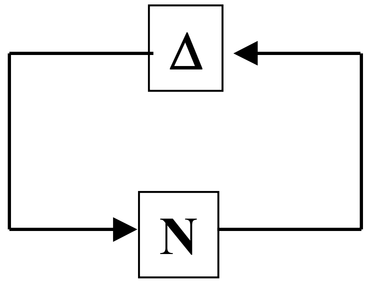
Massaging Block Diagram



- this is known as NΔ-structure, because of the N and the Δ.

Computing N and Δ

- **Using the same familiar techniques we can compute:**



- **with the factors:**

$$\Delta = \begin{bmatrix} \Delta_O & \\ & \Delta_p \end{bmatrix}$$

$$N = \begin{bmatrix} -W_O GK[I+GK]^{-1} & -W_O GK[I+GK]^{-1} \\ W_p[I+GK]^{-1} & W_p[I+GK]^{-1} \end{bmatrix}$$

Equivalence!

- The performance condition:

$$\bar{\sigma}(W_p S) < 1, \qquad \text{for all } G_p$$

- has been transformed into the structured singular value condition:

$$\mu(N) < \frac{1}{\bar{\sigma}(\Delta)}$$

with the "uncertainty" given by,

$$\Delta = \begin{bmatrix} \Delta_O & \\ & \Delta_p \end{bmatrix} \qquad \|\Delta_O\|_\infty \leq 1, \qquad \|\Delta_p\|_\infty \leq 1$$

Comments

- There may be several layers of uncertainty.
- For example, the uncertainty perturbation Δ_O may have further structure:
  - $\Delta_O = \Delta_{full}$ full matrix (i.e. unstructured)
  - $\Delta_O = \Delta_{diag}$ diagonal matrix
- The μ-condition can be applied to a wide variety of RP problem statements.
- See Section 8.10 for general procedure.

Conclusions

- **The structure of model uncertainty is defined during modelling.**
- **If a destabilizing perturbation is small but is not included in the allowable structure, then it is not considered a violation of robust stability.**
- **The structured singular value (μ), is a non-conservative definition of robust stability.**
- **Computing μ typically requires approximation, and the *<u>computation</u>* may introduce conservativeness.**

Conclusions

- The structured singular value μ is a powerful tool for robust control.

- Results were derived for robust stability (RS) and robust performance (RP).