Uncertainty and Robustness for SISO Systems

ELEC 571L – Robust Multivariable Control
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Outline

- Nature of uncertainty (models and signals).
- Physical sources of model uncertainty.
- Mathematical descriptions of model uncertainty.
- Robust stability.
- Robust performance.
Summary of Robustness in Control Systems

- Determine the nominal model $G(s)$ and uncertainty set $G(s) \in \Pi$
- Design controller, $K(s)$.
- Check robust stability (if not RS, return to 2).
- Check robust performance (if not RP, return to 2).

Some controller synthesis techniques (such as $H_\infty$) automate steps 2-4.
Signal Uncertainty versus Model Uncertainty
Suppose we are given an open-loop system model:

\[ g(s) = \frac{1}{s+10} \]

and a constant controller:

\[ k(s) = -9 \]

These make a stable closed-loop with a single pole at \( s = -1 \).
Simulink Model of the Closed-Loop

*Configured for System 2.

Honeywell

Devron Profile Control Solutions
Given that we designed a stabilizing controller for the nominal model (smooth line in red), can we say which of these two systems will perform better in closed-loop?
• System 2 looked better in open-loop, but is far worse in closed-loop.
• Why did this happen?
A 20% change in the gain of the process model destabilized the closed-loop. Bounded additive disturbances have no effect on system stability.
Moral of the story

- Uncertainty is always present in both signals and models.

- Perturbations due to model uncertainty can destabilize a system. Bounded signal uncertainty will not.

- Feedback can create infinite signals. Be careful with it!

- Common sense “analysis” of model uncertainty can be misleading.
Physical Sources of Model Uncertainty
Sources of Model Uncertainty

- Parameters in the linear model
  - identified from noisy input/output data
  - calculated from physical modelling

- Nonlinearities in actuators and sensors
  - actuator/sensor saturation
  - actuator/sensor failure
  - hardware deterioration over time
  - physical systems are inherently nonlinear
Sources of Model Uncertainty

- At high frequencies, we know almost nothing about the process:
  - control and model identification concentrate on low frequencies.

- Deliberate simplification of the model
  - it is easier to design controllers for simple processes than for complicated processes.
Sources of Model Uncertainty

• Uncertainty in the controller (often neglected!)
  
  – deliberate reduction of controller order for simpler implementation
  
  – implementation issues, e.g.
    ♦ finite floating point precision in computers
    ♦ error/limit checking in the controller implementation
  
  – sometimes referred to as fragility (Keel et al, TAC, Aug 97)
Mathematical Descriptions of Model Uncertainty
Parametric Uncertainty

• Most intuitive kind of model uncertainty – uncertainty in the parameters of the linear model.

• For example, consider a first-order transfer function:

\[ \frac{g}{s+a} \]

with parameters

\[ g = 1.0 \pm 0.2 \]
\[ a = 10 \pm 1.3 \]
Parametric Uncertainty

• Write the nominal process model as:

\[ g(s) = \frac{1.0}{(s+10)} \]

• All possible models are given by:

\[ g_p(s) \in \Pi \]

where the uncertainty set is defined by:

\[ \Pi = \{ \frac{g}{(s+a)} : \quad 0.8 \leq g \leq 1.2, \]
\[ \quad 8.7 \leq a \leq 11.3 \} \]
Despite the intuitive nature of parametric uncertainty models, they are not used often.

- The model structure is set, leaving no room for unmodelled dynamics.

- You need to be very sure of the model in order to use such a “high-fidelity” uncertainty structure.

- The mathematics of controller design and analysis is cumbersome in this framework.
• Additive model uncertainty

\[ \Delta_a(s) \rightarrow w_a(s) \rightarrow g(s) \]

where \( \Delta_a(s) \) is any stable transfer function that satisfies \( |\Delta_a(j\omega)| \leq 1 \), for all frequencies \( \omega \).

• we then use transfer function \( w_a(s) \) to model the physical uncertainty (typically high-pass filter).
A Few Words About Delta

• \( \Delta_a(s) \) is different than most transfer functions due to the fact that we assume so little about it.

• We only know that it is stable and \( |\Delta_a(j\omega)| \leq 1 \).

• For example, *any* of the following are permitted:
  
  \[ \Delta_a(s) = +1, +0.5, \text{ etc.} \]
  
  \[ \Delta_a(s) = a / (s+a), \quad \text{for } a > 0 \]
  
  \[ \Delta_a(s) = -1, -0.5, \text{ etc.} \]
  
  \[ \Delta_a(s) = 0 \]
  
  \[ \Delta_a(s) = e^{-\theta s} \]

  ... and infinitely many others!
Additive Model Uncertainty

- The bounds of the possible models can be drawn in the frequency domain.
Additive Model Uncertainty

- This structure permits a family of potential models in a bounded neighbourhood of the nominal model.

Possible models

$|g_p(j\omega)|$

Frequency, [rad/sec]
Multiplicative Model Uncertainty

- configuration is different, but
- each transfer function block has a similar interpretation as with the additive case.
In SISO systems, we can find simple equivalence between additive and multiplicative model uncertainty.

Multiplicative uncertainty seems to be used more often than additive.

Robust stability and performance calculations are simpler with multiplicative than additive uncertainty.
Sometimes a control engineer will deliberately simplify a process model before designing a controller.

\[ g_p(s) = g(s) f(s) \]

where \( f(s) \) is a known transfer function,
Deliberate Model Uncertainty

This may be “covered” by a complex perturbation,

\[ |w_I(j\omega)| \geq |f(j\omega) - 1| \]

if we create \( w_I(s) \) such that
Deliberate Model Uncertainty

• This trick is used when the engineer believes that it will be easier to design a controller for a simple $g(s)$ plus some perturbation (robust control).

• Used when “true” model $g_p(s)$ is complicated.

• Examples include time delays, high-order models, etc.

• Technique should be used carefully, since it “hides” model information from the controller design.
Robust Stability
A closed-loop system with controller $K(s)$ is said to be robustly stable if it is stable for every possible plant in the uncertainty set,

$$G_p(s) \in \Pi$$
Nominal Stability: graphical interpretation

- Remember that (nominal) closed-loop stability requires:

\[ L(j\omega) = K(j\omega)G(j\omega) \] does not encircle \{-1,0\} in complex plane.

![Diagram showing L(j\omega) not encircling \{-1,0\} in the complex plane.](image)
Robust Stability: graphical interpretation

Robust stability requires that

\[ L_p(j\omega) = K(j\omega)G_p(j\omega) \]

does not encircle \{-1,0\} in complex plane,
for any \( G_p(s) \in \Pi \)

How can we guarantee this?
Assume multiplicative uncertainty for the moment:

\[ L_p(s) = G_p(s) K(s) \]
\[ = GK (1 + w_l \Delta_l) = L + w_l L \Delta_l \]

which is a vector equation in the complex plane:
Derivation of Robust Stability

Now, since $\Delta_l(j\omega)$ can have any phase and $|\Delta_l(j\omega)| \leq 1$, then the vector

$$w_1 L \Delta_l$$

defines a disk of radius $|w_1 L|$.

Then the loop transfer function $L_p(j\omega)$ is a vector from $\{0,0\}$ to any point in this disk.
Derivation of Robust Stability

Since RS is satisfied if $L_p(j\omega)$ does not encircle the {-1,0} point, then we need to ensure that the disk never touches {-1,0}.

Diagram shows that the disk never touches {-1,0} if
- $L(j\omega)$ does not encircle {-1,0}, and
- $|w_1L| < |1+L|$, for all $\omega$
Derivation of Robust Stability

The condition,

$$|w| L < |1 + L|$$

is equivalent to,

$$\left| \frac{L(j\omega)}{1 + L(j\omega)} \right| < \left| \frac{1}{w_I(j\omega)} \right|$$

and finally in terms of the “complementary sensitivity”

$$\text{RS for multiplicative uncertainty} \iff \left| T(j\omega) \right| < \left| \frac{1}{w_I(j\omega)} \right|$$
Robust Stability: Comments

• Note the left-hand-side is stated completely in terms of the nominal transfer functions.

• The right-hand-side is stated in terms of the magnitude of the model uncertainty.

• This robust stability condition is \textit{not} conservative. It is both sufficient and necessary to guarantee stability.

• RS conditions for other model uncertainty structures are derived using similar calculations.
Robust Stability: Comments

• Similar RS conditions exist for other model uncertainty structures.

• For example, the RS condition for additive model uncertainty is derived using similar arguments:

\[
\text{RS for additive uncertainty} \iff \left| \frac{K(j\omega)}{1 + L(j\omega)} \right| < \left| \frac{1}{w_A(j\omega)} \right|
\]
Like the majority of robustness tests, these magnitude-based conditions assume that the system is nominally stable.

**RULE #1:** Always check nominal stability (NS), before applying a robust stability (RS) condition.
Robust Performance
A closed-loop system with controller $K(s)$ is said to possess robust performance if the closed-loop satisfies performance specifications (whatever they may be) for every possible plant in the uncertainty set,

$$G_p(s) \in \Pi$$

Remark: In the vast majority (maybe all) of cases, performance specifications will include closed-loop stability.
Let us consider the special case of
- multiplicative uncertainty
- a performance specification of

$$|w_p(j\omega) \cdot S(j\omega)| < 1$$

where $S = (1+L)^{-1} = (1+GK)^{-1}$, and typically we will have the performance weight

$$|w_p(j\omega)| >> 1, \quad \text{for low frequencies } \omega$$
$$|w_p(j\omega)| < 1, \quad \text{for high frequencies } \omega$$
Robust Performance

- First, rewrite the nominal performance specification,

\[
\left| \frac{w_p(j\omega)}{1 + L(j\omega)} \right| < 1 \quad \iff \quad \left| w_p(j\omega) \right| < \left| 1 + L(j\omega) \right|
\]

- which can be represented in a diagram as,
Then to apply the performance specification to all loop gains defined by $L_p(s)$, we can re-use our RS diagram. RP is achieved as long as we keep these disks from touching.
• The disks do not touch if,

\[ | 1 + L(j\omega) | > | w_p(j\omega) | + | w_I(j\omega) L(j\omega) | \]
• The RP condition

$$| 1+L(j\omega) | > | w_p(j\omega) | + | w_I(j\omega) L(j\omega) |$$

can be rewritten as,

$$\left| \frac{w_p(j\omega)}{1+L(j\omega)} \right| + \left| \frac{w_I(j\omega)L(j\omega)}{1+L(j\omega)} \right| < 1$$

• and is typically presented in the form:

$$\left| w_p(j\omega)S(j\omega) \right| + \left| w_I(j\omega)T(j\omega) \right| < 1$$
Robust Performance: Comments

\[ |w_p(j\omega)S(j\omega)| + |w_I(j\omega)T(j\omega)| < 1 \]

- This is a RP condition for a **specific** performance criterion and a **specific** model uncertainty structure.

- RP conditions for other uncertainty structures may be derived.

- Note that satisfying this RP condition automatically includes the RS condition for multiplicative uncertainty \( |w_I(j\omega)T(j\omega)| < 1 \).
In practical robust performance, the size of these disks will depend on the frequency $\omega$.

- Low frequencies, $\omega$:
  - $|w_p(j\omega)|$ large
  - $|w_1(j\omega)|$ small

- High frequencies, $\omega$:
  - $|w_p(j\omega)|$ small
  - $|w_1(j\omega)|$ large
The derivation of this result assumed nominal stability (NS).

**RULE #2:** Always check nominal stability (NS), before applying a robust performance (RP) condition.
Conclusions

• Control systems contain uncertainty in signals and in models.

• Perturbations due to model uncertainty can destabilize a closed-loop. Bounded signal perturbations cannot.

• Robust control concentrates on addressing model uncertainty.
Conclusions

• Complex model perturbations are commonly used to represent model uncertainty.

• Assuming nominal stability (NS):
  – a robust stability (RS) condition was derived in terms of the magnitude of these perturbations.
  – a robust performance (RP) condition was derived in terms of the model uncertainty and the performance specification.

• In practice, the assumption of NS is valid since practical design techniques produce a NS closed-loop.