PERGAMON

Automatica 37 (2001) 221–229

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www.elsevier.com/locate/automatica

Brief Paper

LPV Systems with parameter-varying time delays: analysis and control[☆]

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Received 9 September 1997; revised 24 January 2000; received in final form 23 June 2000

Abstract

In this paper, we address the analysis and state-feedback synthesis problems for linear parameter-varying (LPV) systems with parameter-varying time delays. It is assumed that the state-space data and the time delays are dependent on parameters that are measurable in real-time and vary in a compact set with bounded variation rates. We explore the stability and the induced $\mathscr{L}_2$ norm performance of these systems using parameter-dependent Lyapunov functionals. In addition, the design of parameter-dependent state-feedback controllers that guarantee desired $\mathscr{L}_2$ gain performance is examined. Both analysis and synthesis conditions are formulated in terms of linear matrix inequalities (LMIs) that can be solved via efficient interior-point algorithms.

Keywords: Linear parameter-varying systems; Time-delay systems; H^∞ control; Linear matrix inequalities

1. Introduction

Time delays are often present in engineering systems due to measurement, transmission and transport lags, computational delays, or unmodeled inertias of system components. The stability analysis and control of these systems has been examined extensively in the controls literature using both state-space and frequency domain methods (e.g., see Malek-Zavarei & Jamshidi, 1987; Watanabe, Nobuyama & Kojima, 1996; Dugard & Verriest, 1998, and the references therein). In many engineering systems, the time delays are known functions of variable operating conditions or system parameters that can be measured in real-time. For example, the transport delay in an internal combustion engine is a known function of the engine speed. Similarly, parameter-dependent time delays often appear in many manufacturing and chemical processes, biomedical systems and robotic systems where changes in the system dynamics result in variable delay times. Motivated by the linear parameter-varying (LPV) control theory, in this work the stabilization and the state-feedback control synthesis of such LPV systems that include parameter-dependent time delays is examined. LPV systems are systems that depend on unknown but measurable time-varying parameters, such that the measurement of these parameters provides real-time information on the variations of the plant's characteristics. Hence, it is desirable to design controllers that are scheduled based on this information. The analysis and control of LPV systems has been investigated recently by Packard (1994), Becker and Packard (1994), Apkarian and Gahinet (1995), Wu, Yang, Packard and Becker (1996), and Gahinet, Apkarian and Chilali (1996). These methods provide a systematic gain-scheduling control approach for nonlinear systems (Rugh, 1991; Shamma & Athans, 1990, 1992). The LPV analysis and control synthesis problems can be formulated as linear matrix inequality (LMI) constraints that can be solved using recently developed efficient interior-point optimization algorithms (Boyd, El Ghaoui, Feron & Balakrishnan, 1994; Vandenberghe & Boyd, 1994).

Using the LPV framework, in this work we assume that the state-space system matrices and the time delays are functions of time-varying system parameters that are measured in real-time. We seek to synthesize

☆This paper was not presented at any IFAC meeting. This paper was recommended for publication in revised form by Associate Editor André Tits under the direction of Editor Tamer Basar

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0005-1098/01/$ - see front matter
PII: S0005-1098(00)00156-4

parameter-varying controllers to stabilize the time-delayed LPV system and to provide disturbance attenuation measured in terms of the induced $\mathscr{L}_2$ norm of the system. The proposed approach utilizes parameter-dependent Lyapunov functionals to obtain sufficient conditions for stabilization and induced $\mathscr{L}_2$ norm performance in terms of LMIs. Although the single delay case is considered, the results can be easily extended to treat systems with multiple delays.

The notation to be used is as follows: $\mathbf{R}$ stands for the set of real numbers and $\mathbf{R}_+$ for the non-negative real numbers. $\mathbf{R}^{m\times n}$ is the set of real $m\times n$ matrices. The transpose of a real matrix M is denoted by M^{T} and its orthogonal complements by $M_\perp$. We use $\mathbf{S}^{n\times n}$ to denote real, symmetric $n\times n$ matrices, and $\mathbf{S}_+^{n\times n}$ for positive-definite $n\times n$ matrices. If $M\in\mathbf{S}^{n\times n}$, then $M>0$ ($M\ge 0$) indicates that M is a positive-definite (positive-semi-definite) matrix and $M<0$ ($M\le 0$) denotes a negative-definite (negative-semi-definite) matrix. The matrix norm $\|M\|$ is the maximum singular value of the matrix M, that is $\|M\|:=\bar{\sigma}(M)=[\lambda_{\max}(MM^{\mathrm{T}})]^{1/2}$. For $x\in\mathbf{R}^n$, its norm is defined as $\|x\|:=(x^{\mathrm{T}}x)^{1/2}$. The space of square integrable functions is denoted by $\mathscr{L}_2$, that is, for any $u\in\mathscr{L}_2$, $\|u\|_2:=[\int_0^\infty u^{\mathrm{T}}(t)u(t)\,\mathrm{d}t]^{1/2}$ is finite. The space of continuous functions will be denoted by $\mathscr{C}$ and the corresponding norm is $\|\phi\|=\sup_t\|\phi(t)\|$. In a symmetric block matrix, the expression $(*)$ will be used to denote the submatrices that lie above the diagonal.

2. Analysis of time-delayed LPV systems

We consider the following state-space model of a time-delayed LPV system:

$$\Sigma_d:\quad \dot{x}(t)=A(\rho(t))x(t)+A_h(\rho(t))x(t-h(\rho(t)))+B(\rho(t))d(t), \tag{1}$$

$$e(t)=C(\rho(t))x(t)+C_h(\rho(t))x(t-h(\rho(t)))+D(\rho(t))d(t), \tag{2}$$

$$x(\theta)=\phi(\theta),\quad \theta\in[-h(\rho(0)),0], \tag{3}$$

where $x(t)\in\mathbf{R}^n$ is the state vector, $d(t)\in\mathbf{R}^{n_d}$ is the vector of exogenous inputs, $e(t)\in\mathbf{R}^{n_e}$ in the output vector and h is a differentiable scalar function representing the parameter-varying delay. We assume that the delay is bounded and that the function $t-h(t)$ is monotonically increasing, that is h lies in the set

$$\mathscr{H}:=\{h\in\mathscr{C}(\mathbf{R},\mathbf{R}):\ 0\le h(t)\le H<\infty,\ \dot{h}(t)\le\tau<1,\ \forall t\in\mathbf{R}_+\}.$$

The initial data function ϕ in (3) is a given function in $C([-H,0],\mathbf{R}^n)$. We will use the notation $x_t(\theta)$ to denote $x(t+\theta)$ for $\theta\in[-H,0]$, that is, x_t is the infinite-dimensional state of the delay system.

We assume that the state-space matrices $A(\cdot)$, $A_h(\cdot)$, $B(\cdot)$, $C(\cdot)$, $C_h(\cdot)$, $D(\cdot)$ and the delay $h(\cdot)$ are known continuous functions of a time-varying parameter vector $\rho(\cdot)\in\mathscr{F}_{\mathscr{P}}^{v}$, where $\mathscr{F}_{\mathscr{P}}^{v}$ is the set of allowable parameter trajectories. For our purposes, this set is defined as

$$\mathscr{F}_{\mathscr{P}}^{v}:=\{\rho\in\mathscr{C}(\mathbf{R},\mathbf{R}^s):\ \rho(t)\in\mathscr{P},\ |\dot{\rho}_i(t)|\le v_i,\ i=1,2,\ldots,s,\ \forall t\in\mathbf{R}_+\},$$

where $\mathscr{P}$ is a compact subset of $\mathbf{R}^s$, $\{v_i\}_{i=1}^s$ are non-negative numbers and $v=[v_1\ \cdots\ v_s]^{\mathrm{T}}$, i.e., we consider bounded parameter trajectories with bounded variation rates. Notice that the parametric dependence of the delay on ρ results in a given delay bound H, since ρ is restricted to lie on the given parameter set $\mathscr{P}$. Bounding the rate of variation of the parameter vector ρ allows the use of parameter dependent Lyapunov functionals resulting in less conservative analysis and control synthesis results (Wu et al., 1996; Scherer, 1996; Apkarian & Adams, 1998). The rate of variation should be such that $\dot{h}(t)$ is bounded below 1. It is assumed that at each time instant t the parameter vector $\rho(t)$ is accessible to be measured. We seek to design controllers that are scheduled based on the real-time measurement of ρ.

To examine the stabilization problem consider now the unforced time-delayed LPV system

$$\Sigma_{du}:\quad \dot{x}(t)=A(\rho(t))x(t)+A_h(\rho(t))x(t-h(\rho(t))) \tag{4}$$

with the initial data (3). The following result provides a sufficient condition for asymptotic stability of Σ_{du}. This is an LMI formulation of a corresponding analysis result by Verriest (1994).

Lemma 1. *Consider the unforced delayed LPV system* (4) *with the initial data* (3). *If there exists a continuously differentiable matrix function $P:\mathbf{R}^s\to\mathbf{S}_+^{n\times n}$ and a matrix $Q\in\mathbf{S}_+^{n\times n}$ such that*

$$\begin{bmatrix} A^{\mathrm{T}}(\rho)P(\rho)+P(\rho)A(\rho)+\sum_{i=1}^{s}\left(q_i\dfrac{\partial P}{\partial\rho_i}\right)+Q & (*)\\ A_h^{\mathrm{T}}(\rho)P(\rho) & -\left[1-\sum_{i=1}^{s}\left(q_i\dfrac{\partial h}{\partial\rho_i}\right)\right]Q \end{bmatrix}<0 \tag{5}$$

for all $\rho\in\mathscr{P}$ and $|q_i|\le v_i$ then Σ_{du} is asymptotically stable, that is, the solution $x(\cdot)$ converges to zero as $t\to\infty$ for all $\rho(\cdot)\in\mathscr{F}_{\mathscr{P}}^{v}$.

Proof. Suppose that (5) holds and consider the following Lyapunov–Krasovskii type functional

$$V(x_t,\rho)=x^{\mathrm{T}}(t)P(\rho(t))x(t)+\int_{t-h(\rho(t))}^{t}x^{\mathrm{T}}(\xi)Qx(\xi)\,\mathrm{d}\xi. \tag{6}$$

Let $\bar{\lambda}_P := \max_{\rho\in\mathscr{P}} \lambda_{\max}(P(\rho))$, $\underline{\lambda}_P := \min_{\rho\in\mathscr{P}} \lambda_{\min}(P(\rho))$, and $\bar{\lambda}_Q := \lambda_{\max}(Q)$. Notice that $V(x_t,\rho)$ is bounded from above

$$\begin{aligned} V(x_t,\rho) &\le \|x(t)\|^2\bar{\lambda}_P + \int_{t-H}^{t} x^{\mathrm{T}}(\xi)Qx(\xi)\,\mathrm{d}\xi \\ &\le \|x(t)\|^2\bar{\lambda}_P + \max_{-H\le\theta\le 0} \|x_t(\theta)\|^2 H\bar{\lambda}_{\max}(Q) \\ &\le (\bar{\lambda}_P + H\bar{\lambda}_Q)\|x_t\|^2 \end{aligned}$$

and it is positive-definite since

$$V(x_t,\rho) \ge \underline{\lambda}_P\|x(t)\|^2.$$

Also, for any $\rho(\cdot)\in\mathscr{F}_{\mathscr{P}}^{v}$

$$\begin{aligned} \frac{\mathrm{d}V}{\mathrm{d}t} &= \frac{\mathrm{d}x^{\mathrm{T}}}{\mathrm{d}t}P(\rho(t))x(t) + x^{\mathrm{T}}(t)P(\rho(t))\frac{\mathrm{d}x}{\mathrm{d}t} \\ &\quad + x^{\mathrm{T}}(t)\frac{\mathrm{d}P}{\mathrm{d}t}x(t) + x^{\mathrm{T}}(t)Qx(t) \\ &\quad - \left(1-\frac{\mathrm{d}h}{\mathrm{d}t}\right)x^{\mathrm{T}}(t-h(\rho(t)))Qx(t-h(\rho(t))) \\ &= [x^{\mathrm{T}}(t)\; x^{\mathrm{T}}(t-h(\rho(t)))] \\ &\quad \times \begin{bmatrix} A^{\mathrm{T}}P + PA + \dot{P} + Q & (*) \\ A_h^{\mathrm{T}}P & (-1+\dot{h})Q \end{bmatrix} \\ &\quad \times \begin{bmatrix} x(t) \\ x(t-h(\rho(t))) \end{bmatrix} \\ &\le 0, \end{aligned}$$

where the last inequality follows from (5). Hence, $V(x_t,\rho)$ is a Lyapunov functional and the system Σ_{du} is asymptotically stable (Driver, 1977; Kolmanovskii & Shaikhet, 1996). □

An equivalent result based on Lyapunov–Krasovskii functionals has been derived by Verriest (1994) to examine the stability of general linear time-varying (LTV) systems resulting in a *Riccati differential equation* stability conditions. In our case, the proposed LMI formulation of this result will lead to computationally attractive analysis and synthesis conditions for the time-delayed LPV system (4). It is noted that the proposed Lyapunov–Krasovskii based analysis is a conservative one; however, it provides easily computable results based on convex optimization. The restricted parameter dependency of the Lyapunov–Krasovskii functional, with Q constant, is selected to provide this computational advantage.

The following result provides a sufficient condition for the induced $\mathscr{L}_2$ gain performance of the forced time-delayed LPV system (1), (2).

Theorem 2. *Consider the delayed system Σ_d (1), (2) with initial data $\phi \equiv 0$. If there exists a continuously differentiable matrix function $P:\mathbf{R}^s \to \mathbf{S}_+^{n\times n}$ and a matrix $Q\in\mathbf{S}_+^{n\times n}$, such that*

$$\begin{bmatrix} A^{\mathrm{T}}(\rho)P(\rho) + P(\rho)A(\rho) + \sum_{i=1}^{s} \pm\left(v_i\frac{\partial P}{\partial \rho_i}\right) + Q & (*) & (*) & (*) \\ A_h^{\mathrm{T}}(\rho)P(\rho) & -\left[1 - \sum_{i=1}^{s} \pm\left(v_i\frac{\partial h}{\partial\rho_i}\right)\right] Q & (*) & (*) \\ B^{\mathrm{T}}(\rho)P(\rho) & 0 & -\gamma I & (*) \\ C(\rho) & C_h(\rho) & D(\rho) & -\gamma I \end{bmatrix} < 0 \qquad (7)$$

for all $\rho\in\mathscr{P}$ then Σ_d is asymptotically stable and has induced $\mathscr{L}_2$ norm less than γ.

Remark 1. The notation $\sum_{i=1}^{s} \pm(\cdot)$ in (7) is used to indicate that every combination of $+(\cdot)$ and $-(\cdot)$ should be included in the inequality. That is, inequality (7) actually represents 2^s different inequalities that correspond to the 2^s different combinations in the summation.

Proof of Theorem 2. From condition (7), it follows that

$$\begin{bmatrix} A^{\mathrm{T}}(\rho(t))P(\rho(t)) + P(\rho(t))A(\rho(t)) + \frac{\mathrm{d}P}{\mathrm{d}t} + Q & (*) & (*) & (*) \\ A_h^{\mathrm{T}}(\rho(t))P(\rho(t)) & -(1-\frac{\mathrm{d}h}{\mathrm{d}t})Q & (*) & (*) \\ B^{\mathrm{T}}(\rho(t))P(\rho(t)) & 0 & -\gamma I & (*) \\ C(\rho(t)) & C_h(\rho(t)) & D(\rho(t)) & -\gamma I \end{bmatrix} < 0 \qquad (8)$$

for any admissible trajectory $\rho(\cdot)\in\mathscr{F}_{\mathscr{P}}^{v}$. From the top 2×2 sub-matrix and Lemma 1, it follows that the delayed LPV system Σ_d is asymptotically stable. Consider again the parameter-dependent Lyapunov–Krasovskii functional (6) and notice that

$$
\begin{aligned}
\frac{\mathrm{d}V}{\mathrm{d}t} + \gamma^{-1}e^{\mathrm{T}}(t)e(t) - \gamma d^{\mathrm{T}}(t)d(t) &= \frac{\mathrm{d}x^{\mathrm{T}}}{\mathrm{d}t}P(\rho(t))x(t) + x^{\mathrm{T}}(t)P(\rho(t))\frac{\mathrm{d}x}{\mathrm{d}t} + x^{\mathrm{T}}(t)\frac{\mathrm{d}P}{\mathrm{d}t}x(t) \\
&\quad + x^{\mathrm{T}}(t)Qx(t) - \left(1 - \frac{\mathrm{d}h}{\mathrm{d}t}\right)x^{\mathrm{T}}(t - h(\rho(t)))Qx(t - h(\rho(t))) + \gamma^{-1}\mathrm{e}^{\mathrm{T}}(t)e(t) - \gamma d^{\mathrm{T}}(t)d(t) \\
&= [x^{\mathrm{T}}(t)\ x^{\mathrm{T}}(t - h(\rho(t)))d^{\mathrm{T}}(t)]
\end{aligned}
$$

$$
\times \begin{bmatrix} A^{\mathrm{T}}P + PA + \dot{P} + Q + \gamma^{-1}C^{\mathrm{T}}C & (*) & (*) \\ A_h^{\mathrm{T}}P + \gamma^{-1}C_h^{\mathrm{T}}C & (-1 + \dot{h})Q + \gamma^{-1}C_h^{\mathrm{T}}C_h & (*) \\ B^{\mathrm{T}}P + \gamma^{-1}D^{\mathrm{T}}C & \gamma^{-1}D^{\mathrm{T}}C_h & -\gamma I + \gamma^{-1}D^{\mathrm{T}}D \end{bmatrix} \begin{bmatrix} x(t) \\ x(t - h(\rho(t))) \\ d(t) \end{bmatrix}
$$

$$\leq 0.$$

The last inequality follows from the Schur complement of inequality (8) (see Skelton, Iwasaki & Grigoriadis, 1998). Integrating both sides of the above inequality from 0 to ∞, and noting that $V(\infty) = 0$ due to the asymptotic stability of the delayed LPV system, we obtain

$$\|e\|_2^2 \leq \gamma^2\|d\|_2^2,$$

which implies that the induced $\mathscr{L}_2$ norm of Σ_d from d to e is less than γ. □

Notice that the LMI conditions (5) and (7) correspond to infinite-dimensional convex problems due to their parametric dependence. To obtain a finite-dimensional optimization problem, the parameter-dependent matrix function $P(\rho)$ that appears in the stability and performance conditions (5) and (7) can be approximated using a finite set of basis functions (Scherer, 1996; Wu et al., 1996; Apkarian & Adams, 1998). Hence, by choosing appropriate basis functions $\{f_j(\rho)\}_{j=1}^{n_f}$ such that

$$P(\rho) = \sum_{j=1}^{n_f} f_j(\rho)P_j, \quad P_j = P_j^{\mathrm{T}}, \tag{9}$$

the analysis condition in Theorem 2 can be approximated as follows

Corollary 3. *Consider the parametrization* (9) *of the matrix function* $P(\rho)$. *Then, the delayed LPV system* Σ_d *is asymptotically stable and has induced* $\mathscr{L}_2$ *norm less than* γ *if there exist symmetric matrices* $\{P_j\}_{j=1}^{n_f}$ *and* $Q > 0$ *satisfying the LMI constraints*

$$P(\rho) = \sum_{j=1}^{n_f} f_j(\rho)P_j > 0 \tag{10}$$

$$
\begin{bmatrix} A^{\mathrm{T}}(\rho)P(\rho) + P(\rho)A(\rho) + \sum_{i=1}^{s} \pm\left(v_i \sum_{j=1}^{n_f} \frac{\partial f_j}{\partial \rho_i}P_j\right) + Q & (*) & (*) & (*) \\ A_h^{\mathrm{T}}(\rho)P(\rho) & -\left[1 - \sum_{i=1}^{s} \pm\left(v_i\frac{\partial h}{\partial \rho_i}\right)\right]Q & (*) & (*) \\ B^{\mathrm{T}}(\rho)P(\rho) & 0 & -\gamma I & (*) \\ C(\rho) & C_h(\rho) & D(\rho) & -\gamma I \end{bmatrix} < 0 \tag{11}
$$

for all $\rho \in \mathscr{P}$.

Conditions (10) and (11) represent $(2^s + 1)$ LMIs on the matrix variables $\{P_j\}_{j=1}^{n_f}$ and Q, and the scalar γ. To eliminate the dependence on the parameter vector ρ, a finite 6g-ridding $\{\rho_k\}_{k=1}^{L}$ of the parameter space $\mathscr{P}$ can be used resulting in $L^s(2^s + 1)$ finite-dimensional LMIs that can be solved numerically using convex optimization techniques (Gahinet, Nemirovskii, Laub & Chilali, 1995). The basis functions f_j, $j = 1, \ldots, n_f$ should be selected by the designer. Notice that obviously the proposed basis approximation restricts the allowable range of Lyapunov functions. A detailed discussion on the gridding technique and the selection of appropriate basis functions for parameter dependent LMIs can be found in Apkarian and Adams (1998).

Remark 2. It can be shown that in the absence of a delay, conditions (7) result in the $\mathscr{L}_2$ gain bounding conditions for non-delayed LPV systems obtained in Wu et al. (1996).

Remark 3. Using a similar approach, LMI-based stability and induced $\mathscr{L}_2$ norm performance analysis results for a time-delayed LPV system

$$
\begin{aligned}
\dot{x}(t) &= A(\rho(t))x(t) + \sum_{i=1}^{k} A_{hi}(\rho(t))x(t - h_i(\rho(t))) \\
&\quad + B(\rho(t))d(t)
\end{aligned}
$$

with multiple delays h_i, $i = 1, \ldots, k$ such that $0 \leq h_i < \infty$ and $\mathrm{d}h_i/\mathrm{d}t < 1$ can be easily derived using the following functional:

$$V(x_t, \rho) = x^{\mathrm{T}}(t)P(\rho(t))x(t) + \int_{t - h_1(\rho(t))}^{t} x^{\mathrm{T}}(\xi)Q_1 x(\xi)\,\mathrm{d}\xi$$

$$+\int_{t-h_2(\rho(t))}^{t-h_1(\rho(t))} x^{\mathrm{T}}(\xi)Q_2x(\xi)\,\mathrm{d}\xi + \cdots$$
$$+\int_{t-h_k(\rho(t))}^{t-h_{k-1}(\rho(t))} x^{\mathrm{T}}(\xi)Q_kx(\xi)\,\mathrm{d}\xi.$$

3. State-feedback control of time-delayed LPV systems

In this section, the analysis results presented in the previous section are used to design state-feedback controllers for LPV systems with parameter-dependent state and input time delays. We seek to design state-feedback gains that are scheduled based on the real-time measurement of the parameter vector ρ and guarantee prescribed induced $\mathscr{L}_2$ norm performance levels for the closed-loop system.

Consider the following time-delayed LPV system:

$$\begin{aligned}\dot{x}(t) &= A(\rho(t))x(t) + A_h(\rho(t))x(t-h(\rho(t))) + B_1(\rho(t))d(t)\\ &\quad + B_2(\rho(t))u(t) + B_{2h}(\rho(t))u(t-h(\rho(t))), \end{aligned}\tag{12}$$

$$\begin{aligned}e(t) &= C_1(\rho(t))x(t) + C_{1h}(\rho(t))x(t-h(\rho(t)))\\ &\quad + D_{12}(\rho(t))u(t) + D_{12h}(\rho(t))u(t-h(\rho(t))),\end{aligned}\tag{13}$$

where $\rho\in\mathscr{F}^{v}_{\mathscr{P}}$, $x(t)\in\mathbf{R}^{n}$, $d(t)\in\mathbf{R}^{n_d}$, $e(t)\in\mathbf{R}^{n_e}$ and $u(t)\in\mathbf{R}^{n_u}$. We assume that all state-space data and the delay h are continuous functions of the parameter ρ. We will assume that for all $\rho\in\mathscr{P}$

(A1) $D_{12}(\rho)$ has full column rank,
(A2) $(A(\cdot),B_2(\cdot))$ is asymptotically stabilizable, that is, there exists a parameter-dependent state-feedback controller $u(t)=F(\rho(t))x(t)$, such that the closed-loop LPV system is asymptotically stable.

Note that assumption (A1) can be easily relaxed, but we will use it to simplify the presentation. Based on these assumptions, we consider the following normalized structure for matrices $C_1(\rho), C_{1h}(\rho)$ and $D_{12}(\rho)$:

$$C_1=\begin{bmatrix}C_{11}(\rho)\\ C_{12}(\rho)\end{bmatrix},\quad C_{1h}=\begin{bmatrix}C_{11h}(\rho)\\ C_{12h}(\rho)\end{bmatrix},\quad D_{12}(\rho)=\begin{bmatrix}0\\ I\end{bmatrix}.$$

Also, the case where there is a feed-through term $D_{11}(\rho(t))d(t)$ in the output equation (13) can be easily handled, but it will not be discussed here for simplicity.

3.1. LPV systems with state delay

We first consider the state-delayed LPV control synthesis problem, that is, we assume that $B_{2h}(\rho)=0$ and $D_{12h}(\rho)=0$. We seek to design a parameter-dependent state-feedback controller

$$u(t)=F(\rho(t),\dot\rho(t))x(t),\tag{14}$$

such that the closed-loop system is asymptotically stable and has induced $\mathscr{L}_2$ norm less than a pre-specified bound γ. Using the state-feedback control law (14) the closed-loop system becomes

$$\begin{aligned}\dot{x}(t) &= A_F(\rho(t),\dot\rho(t))x(t) + A_h(\rho(t))x(t-h(\rho(t)))\\ &\quad + B_1(\rho(t))d(t),\end{aligned}\tag{15}$$

$$e(t)=C_F(\rho(t),\dot\rho(t))x(t) + C_{1h}(\rho(t))x(t-h(\rho(t))),\tag{16}$$

where

$$A_F(\rho,\dot\rho):=A(\rho)+B_2(\rho)F(\rho,\dot\rho),$$
$$C_F(\rho,\dot\rho):=C_1(\rho)+D_{12}(\rho)F(\rho,\dot\rho).$$

The following result provides conditions for the closed-loop system (15), (16) to be asymptotically stable and have induced $\mathscr{L}_2$ norm less than γ.

Theorem 4. *Consider the time-delayed LPV system* (12), (13) *with* $B_{2h}(\rho)=0$ *and* $D_{12h}(\rho)=0$. *There exists a parameter-dependent controller* (14) *such that the closed-loop system is asymptotically stable and has induced* $\mathscr{L}_2$ *norm less than* γ *if there exists a continuously differentiable matrix function* $R:\mathbf{R}^s\to\mathbf{S}_+^{n\times n}$ *and a matrix* $S\in\mathbf{S}_+^{n\times n}$, *such that for all* $\rho\in\mathscr{P}$

$$\begin{bmatrix}\left\{\begin{matrix}R(\rho)\hat{A}^{\mathrm{T}}(\rho)+\hat{A}(\rho)R(\rho)-\sum_{i=1}^{s}\pm\left(v_i\dfrac{\partial R}{\partial\rho_i}\right)\\ +\psi\hat{A}_h(\rho)S\hat{A}_h^{\mathrm{T}}(\rho)-\gamma B_2(\rho)B_2^{\mathrm{T}}(\rho)\end{matrix}\right\} & (*) & (*) & (*)\\ R(\rho) & -S & (*) & (*)\\ B_1^{\mathrm{T}}(\rho) & 0 & -\gamma I & (*)\\ C_{11}(\rho)R(\rho)+\psi C_{11h}(\rho)S\hat{A}_h^{\mathrm{T}}(\rho) & 0 & 0 & -\gamma I+\psi C_{11h}(\rho)SC_{11h}^{\mathrm{T}}(\rho)\end{bmatrix}<0,\tag{17}$$

$$-\gamma I+\psi C_{1h}(\rho)SC_{1h}^{\mathrm{T}}(\rho)<0,\tag{18}$$

where

$$\psi=\left[1-\sum_{i=1}^{s}\pm\left(v_i\frac{\partial h}{\partial\rho_i}\right)\right]^{-1},$$

$$\hat{A}(\rho):=A(\rho)-B_2(\rho)C_{12}(\rho),$$

$$\hat{A}_h(\rho):=A_h(\rho)-B_2(\rho)C_{12h}(\rho).$$

Moreover, one such state-feedback control law that provides a guaranteed $\mathscr{L}_2$ *gain performance* γ *is given by*

$$F(\rho(t),\dot{\rho}(t)) = -F_1^{-1}(\rho(t),\dot{\rho}(t))F_2(\rho(t),\dot{\rho}(t)), \tag{19}$$

where

$$F_1(\rho,\dot{\rho}) = I + \gamma^{-1}C_{12h}(\rho) \times \left[\left(1 - \sum_{i=1}^{s}\dot{\rho}_i\frac{\partial h}{\partial \rho_i}\right)S^{-1} - \gamma^{-1}C_{1h}^{\mathrm{T}}(\rho)C_{1h}(\rho)\right]^{-1} C_{12h}^{\mathrm{T}}(\rho)$$

and

$$F_2(\rho,\dot{\rho}) = \gamma B_2^{\mathrm{T}}(\rho)R^{-1}(\rho) + C_{12}(\rho) + C_{12h}(\rho) \times \left[\left(1 - \sum_{i=1}^{s}\dot{\rho}_i\frac{\partial h}{\partial \rho_i}\right)S^{-1} - \gamma^{-1}C_{1h}^{\mathrm{T}}(\rho)C_{1h}(\rho)\right]^{-1} \times [A_h^{\mathrm{T}}(\rho)R^{-1}(\rho) + \gamma^{-1}C_{1h}^{\mathrm{T}}(\rho)C_1(\rho)].$$

Proof. Choosing $F(\rho,\dot{\rho})$ as in (19), $P(\rho) = R^{-1}(\rho)$ and $Q = S^{-1}$ it can be easily verified that conditions (17) and (18) are equivalent to the solvability conditions (Skelton et al., 1998)

$$U_\perp^{\mathrm{T}}(\rho)W(\rho)U_\perp(\rho) < 0 \quad \text{and} \quad V_\perp(\rho)W(\rho)V_\perp^{\mathrm{T}}(\rho) < 0$$

for the induced $\mathscr{L}_2$ norm inequality (7), where

$$W(\rho) = \begin{bmatrix} \left\{\begin{matrix} R(\rho)A^{\mathrm{T}}(\rho) + A(\rho)R(\rho) - \sum_{i=1}^{s} \pm\left(v_i\frac{\partial R}{\partial \rho_i}\right) \\ + \psi A_h(\rho)SA_h^{\mathrm{T}}(\rho) \end{matrix}\right\} & (*) & (*) & (*) \\ C_1(\rho)R(\rho) + \psi C_{1h}(\rho)SA_h^{\mathrm{T}}(\rho) & -\gamma I + \psi C_{1h}(\rho)SC_{1h}^{\mathrm{T}}(\rho) & (*) & (*) \\ B_1^{\mathrm{T}}(\rho) & 0 & -\gamma I & (*) \\ R(\rho) & 0 & 0 & -S \end{bmatrix},$$

$$U(\rho) = \begin{bmatrix} B_2(\rho) \\ D_{12}(\rho) \\ 0 \\ 0 \end{bmatrix}, \quad V(\rho) = [R(\rho) \quad 0 \quad 0 \quad 0].$$

Hence, based on Theorem 2, the closed-loop system is asymptotically stable and has induced $\mathscr{L}_2$ norm less than γ. □

Remark 4. It can be easily shown that (17) and (18) are also necessary for the time-delayed closed-loop LPV system to satisfy the condition of Theorem 2.

The controller construction consists of solving a finite-dimensional approximation of the LMIs (17) and (18) for $R(\rho)$ and S as discussed in Section 2, and implementing the parameter-dependent state-feedback control gain given by (19). The optimal bounding $\mathscr{L}_2$ gain performance based on the proposed conditions can be obtained by minimizing the scalar γ with respect to the above LMIs.

3.2. LPV systems with input delay

Next, we consider the control synthesis problem for the input-delayed LPV system (12), (13), that is, we assume that $A_h(\rho) = 0$ and $C_{1h}(\rho) = 0$. However, the analysis results developed in Section 2 cannot be used directly for input-delayed LPV control. To address this problem, we introduce an artificial dynamic feedback control law $u_a(t) \in \mathbf{R}^{n_u}$ as follows:

$$u(s) = (sI + \Lambda)^{-1}Ku_a(s),$$

where K is a non-singular gain matrix and $\Lambda > 0$ is a parameter matrix that can be selected as the bandwidth of the actuators. Then, by defining the new state vector $x_a^{\mathrm{T}} = [x^{\mathrm{T}} \; u^{\mathrm{T}}]$, we obtain a state-delayed LPV system as follows:

$$\dot{x}_a(t) = \begin{bmatrix} A(\rho(t)) & B_2(\rho(t)) \\ 0 & -\Lambda \end{bmatrix}x_a(t) + \begin{bmatrix} 0 & B_{2h}(\rho(t)) \\ 0 & 0 \end{bmatrix}x_a(t - h(\rho(t))) + \begin{bmatrix} B_1(\rho(t)) \\ 0 \end{bmatrix}d(t) + \begin{bmatrix} 0 \\ K \end{bmatrix}u_a(t) \tag{20}$$

$$e(t) = [C_1(\rho(t)) \quad D_{12}(\rho(t))]x_a(t) + [0 \quad D_{12h}(\rho(t))]x_a(t - h(\rho(t))). \tag{21}$$

It can be easily verified that the stabilization assumption (A2) holds for this augmented LPV plant. Assumption (A1) is not satisfied, but this can be remedied by slightly perturbing the output vector $e(t)$ using a small input term. Now, Theorem 4 can be applied to this new "state-delayed" LPV system (20), (21). Suppose that the resulting state-feedback control law is given by

$$u_a(t) = F_a(\rho(t))x_a(t) := [F_x(\rho(t)) \quad F_u(\rho(t))]\begin{bmatrix} x(t) \\ u(t) \end{bmatrix}.$$

Then, the following parameter-dependent controller provides induced $\mathscr{L}_2$ norm performance γ to the original input-delayed LPV system

$$\dot{u}(t) = KF_x(\rho(t))x(t) + (KF_u(\rho(t)) - \Lambda)u(t).$$

Therefore, in the proposed approximate approach the control input u is governed by a differential equation instead of a static feedback gain. This approach can be extended to treat control synthesis problems for LPV systems with both state and input delays.

4. Numerical example

Consider the following linear time-varying state-delayed system adopted from Mahmoud and Al-Muthairi (1994):

$$\dot{x}(t) = \begin{bmatrix} 0 & 1+\phi\sin t \\ -2 & -3+\delta\sin t \end{bmatrix} x(t) + \begin{bmatrix} \phi\sin t & 0.1 \\ -0.2+\delta\sin t & -0.3 \end{bmatrix} x(t-\mu|\cos(\omega t)|) + \begin{bmatrix} 0.2 \\ 0.2 \end{bmatrix} d(t) + \begin{bmatrix} \phi\sin t \\ 0.1+\delta\sin t \end{bmatrix} u(t), \quad (22)$$

$$e(t) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t), \quad (23)$$

where $\phi = 0.2$, $\delta = 0.1$, $\mu = 0.09$ and $\omega = 5$. To validate our proposed time-delayed LPV design methodology, we will assume that the sine and cosine terms in the above model correspond to systems parameters whose functional representation is not known a priori, but they can be measured in real time. Hence, we define $\rho_1(t) := \sin t$ and $\rho_2(t) := |\cos(\omega t)|$ and the original system is formulated as a state-delayed LPV system as follows:

$$\dot{x}(t) = \begin{bmatrix} 0 & 1+\phi\rho_1(t) \\ -2 & -3+\delta\rho_1(t) \end{bmatrix} x(t) + \begin{bmatrix} \phi\rho_1(t) & 0.1 \\ -0.2+\delta\rho_1(t) & -0.3 \end{bmatrix} x(t-\mu\rho_2(t)) + \begin{bmatrix} 0.2 \\ 0.2 \end{bmatrix} d(t) + \begin{bmatrix} \phi\rho_1(t) \\ 0.1+\delta\rho_1(t) \end{bmatrix} u(t),$$

$$e(t) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t).$$

The parameter space is $[-1\ 1] \times [0\ 1]$. The time delay $h(t) = \mu\rho_2(t)$ is varying from 0 to 0.09 and the condition $\mathrm{d}h/\mathrm{d}t < 1$ holds except for a countable number of points. Moreover, $|\mathrm{d}\rho_1/\mathrm{d}t| \le 1$ and $|\mathrm{d}\rho_2/\mathrm{d}t| \le 5$. A controller that only depends on the real-time measurement of $\rho(t) = [\rho_1(t)\ \rho_2(t)]^{\mathrm{T}}$, thus not on future values of $\rho(t)$, is sought here.

To solve the synthesis problem, we pick three basis functions in expansion (10) as follows:

$$f_1(\rho) = 1, \quad f_2(\rho) = \rho_1, \quad f_3(\rho) = \rho_2. \quad (24)$$

Gridding the parameter space uniformly using a 9×9 grid, we obtain from Theorem 4 an induced $\mathscr{L}_2$ norm performance bound $\gamma_{\mathbf{LPV}} = 2.38$ with the following parameter matrices

$$R_1 = \begin{bmatrix} 0.04119 & -0.02120 \\ -0.02120 & 0.06044 \end{bmatrix},$$

$$R_2 = \begin{bmatrix} 1.6821\times10^{-6} & -3.4615\times10^{-5} \\ -3.4615\times10^{-5} & 2.2543\times10^{-5} \end{bmatrix},$$

$$R_3 = \begin{bmatrix} 2.6125\times10^{-4} & 9.2482\times10^{-4} \\ 9.2482\times10^{-4} & 4.8084\times10^{-3} \end{bmatrix}.$$

Hence, the parameter-dependent state-feedback control law given by (19) results in

$$F(\rho) = -2.38 \times [\phi\rho_1\ \ 0.1+\delta\rho_1] (R_1 + \rho_1 R_2 + \rho_2 R_3)^{-1}. \quad (25)$$

For an initial condition $(x_1(0), x_2(0)) = (2, -1)$ and a unit step disturbance $d(t)$, we simulate the closed-loop behavior of the system using the LPV state-feedback control law (25). The states and control input profiles are shown in Figs. 1 and 2. Note that both states x_1, x_2 converge to zero very rapidly.

The effect of the maximum delay magnitude μ on the attainable induced $\mathscr{L}_2$ norm performance bound γ is shown in Fig. 3. Notice that as expected, increase of the delay magnitude results in performance deterioration.

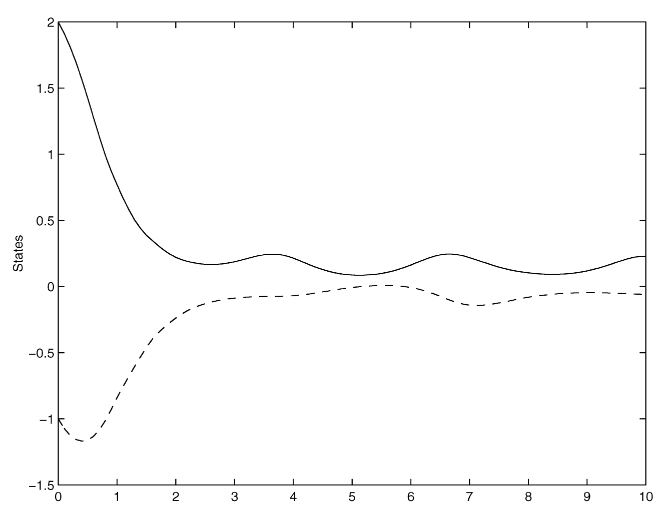

Fig. 1. System response: x_1 (solid) and x_2 (dashed).

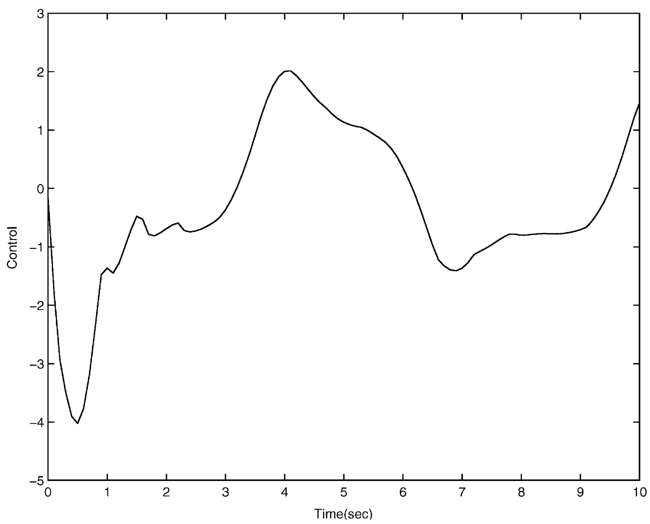

Fig. 2. Control input.

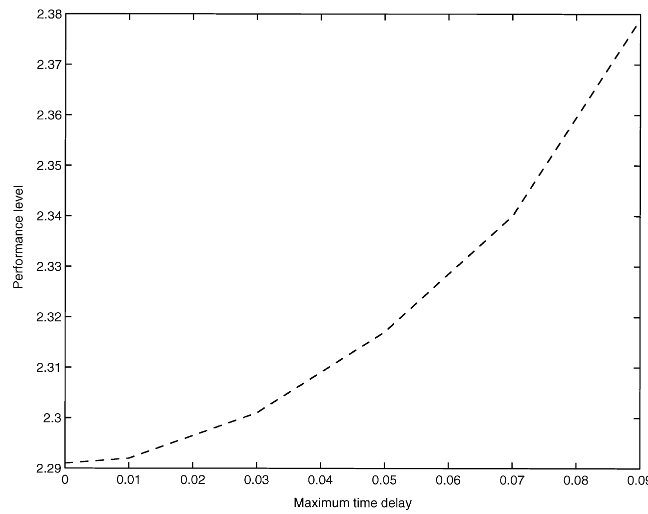

Fig. 3. Maximum delay magnitude vs. induced $\mathscr{L}_2$ norm performance.

Next, we consider the following input-delayed LPV system by slightly modifying the previous model

$$\dot{x}(t) = \begin{bmatrix} 0 & 1+\phi\rho_1(t) \\ -2 & -3+\delta\rho_1(t) \end{bmatrix} x(t) + \begin{bmatrix} 0.2 \\ 0.2 \end{bmatrix} d(t) + \begin{bmatrix} \phi\rho_1(t) \\ 0.1+\delta\rho_1(t) \end{bmatrix} u(t) + \begin{bmatrix} 0.1 \\ -0.3 \end{bmatrix} u(t-\mu\rho_2(t)), \quad (26)$$

$$e(t) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t). \quad (27)$$

where the scheduling parameters have been selected as before. The input-delayed LPV system (26), (27) can be transformed to a state-delayed dynamic model by the state augmentation proposed in Section 3. Choosing the parameter values $K = 10^{-3}$ and $\Lambda = 1$, the converted state-delayed LPV system is

$$\begin{bmatrix} \dot{x}(t) \\ \dot{u}(t) \end{bmatrix} = \begin{bmatrix} 0 & 1+\phi\rho_1(t) & \phi\rho_1(t) \\ -2 & -3+\delta\rho_1(t) & 0.1+\delta\rho_1(t) \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} + \begin{bmatrix} 0.1 \\ -0.3 \\ 0 \end{bmatrix} u(t-\mu\rho_2(t)) + \begin{bmatrix} 0.2 \\ 0.2 \\ 0 \end{bmatrix} d(t) + \begin{bmatrix} 0 \\ 0 \\ K \end{bmatrix} u_a(t),$$

$$e(t) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 10^{-3} \end{bmatrix} u_a(t).$$

Note that we have penalized the artificial control u_a by a small quantity in the output equation to avoid singularity problems. Using Theorem 4 and the same basis functions as before, we obtain a performance level $\gamma_{\mathrm{LPV}} = 2.30$. In this case, our parameter-dependent controller is given by

$$\dot{u}(t) = -\{[0\ 0\ 2.30] \times (R_1 + \rho_1(t)R_2 + \rho_2(t)R_3)^{-1} + [0 \quad 0 \quad 2]\} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix},$$

where

$$R_1 = \begin{bmatrix} 0.03594 & -0.02023 & 0.01661 \\ -0.02023 & 0.06441 & -0.04863 \\ 0.01661 & -0.04863 & 0.2183 \end{bmatrix},$$

$$R_2 = \begin{bmatrix} -3.1343\times 10^{-4} & -2.0156\times 10^{-4} & -8.9913\times 10^{-3} \\ -2.0156\times 10^{-4} & 6.2186\times 10^{-4} & -5.6514\times 10^{-3} \\ -8.9913\times 10^{-3} & -5.6514\times 10^{-3} & 3.4805\times 10^{-3} \end{bmatrix},$$

$$R_3 = \begin{bmatrix} 1.9984\times 10^{-4} & -5.8750\times 10^{-4} & 5.4119\times 10^{-5} \\ -5.8750\times 10^{-4} & 1.7304\times 10^{-3} & -9.7013\times 10^{-5} \\ 5.4119\times 10^{-5} & -9.7013\times 10^{-5} & 2.8978\times 10^{-3} \end{bmatrix}.$$

5. Conclusions

In this paper, the analysis and state-feedback control synthesis problems for LPV systems with parameter-dependent state delays are addressed. The corresponding analysis and synthesis conditions for stabilization and induced $\mathscr{L}_2$ norm performance are expressed in terms of LMIs that can be solved efficiently using recently developed interior-point algorithms. In addition, we considered the state-feedback control synthesis problem for LPV systems with input delays by augmenting the system dynamics and transforming the problem to a state-delayed control problem. However, in this case a dynamic state-feedback controller is required. These results provide a systematic procedure to address parameter-dependent time-delays in a gain-scheduling control design framework for non-linear systems.

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