5 Signal Design for Bandlimited Channels

- So far, we have not imposed any bandwidth constraints on the transmitted passband signal, or equivalently, on the transmitted baseband signal

\[ s_b(t) = \sum_{k=-\infty}^{\infty} I[k] g_T(t - kT), \]

where we assume a linear modulation (PAM, PSK, or QAM), and \( I[k] \) and \( g_T(t) \) denote the transmit symbols and the transmit pulse, respectively.

- In practice, however, due to the properties of the transmission medium (e.g. cable or multipath propagation in wireless), the underlying transmission channel is bandlimited. If the bandlimited character of the channel is not taken into account in the signal design at the transmitter, the received signal will be (linearly) distorted.

- In this chapter, we design transmit pulse shapes \( g_T(t) \) that guarantee that \( s_b(t) \) occupies only a certain finite bandwidth \( W \). At the same time, these pulse shapes allow ML symbol–by–symbol detection at the receiver.
5.1 Characterization of Bandlimited Channels

- Many communication channels can be modeled as *linear filters* with (equivalent baseband) impulse response $c(t)$ and frequency response $C(f) = \mathcal{F}\{c(t)\}$.

\[ y_b(t) = \int_{-\infty}^{\infty} s_b(\tau) c(t - \tau) d\tau + z(t), \]

where $z(t)$ denotes complex Gaussian noise with power spectral density $N_0$.

- We assume that the channel is ideally bandlimited, i.e.,

\[ C(f) = 0, \quad |f| > W. \]
Within the interval $|f| \leq W$, the channel is characterized by

$$C'(f) = |C(f)| e^{j\Theta(f)}$$

with amplitude response $|C(f)|$ and phase response $\Theta(f)$.

If $S_b(f) = \mathcal{F}\{s_b(t)\}$ is non-zero only in the interval $|f| \leq W$, then $s_b(t)$ is not distorted if and only if

1. $|C(f)| = A$, i.e., the amplitude response is constant for $|f| \leq W$.

2. $\Theta(f) = -2\pi f \tau_0$, i.e., the phase response is linear in $f$, where $\tau_0$ is the delay.

In this case, the received signal is given by

$$y_b(t) = A s_b(t - \tau_0) + z(t).$$

As far as $s_b(t)$ is concerned, the impulse response of the channel can be modeled as

$$c(t) = A \delta(t - \tau_0).$$

If the above conditions are not fulfilled, the channel linearly distorts the transmitted signal and an equalizer is necessary at the receiver.
5.2 Signal Design for Bandlimited Channels

- We assume an ideal, bandlimited channel, i.e.,
  \[ C(f) = \begin{cases} 
  1, & |f| \leq W \\
  0, & |f| > W 
  \end{cases} \]

- The transmit signal is
  \[ s_b(t) = \sum_{k=-\infty}^{\infty} I[k] g_T(t - kT). \]

- In order to avoid distortion, we assume
  \[ G_T(f) = \mathcal{F}\{g_T(t)\} = 0, \quad |f| > W. \]
  This implies that the received signal is given by
  \[ y_b(t) = \int_{-\infty}^{\infty} s_b(\tau)c(t - \tau) \, d\tau + z(t) \]
  \[ = s_b(t) + z(t) \]
  \[ = \sum_{k=-\infty}^{\infty} I[k] g_T(t - kT) + z(t). \]

- In order to limit the noise power in the *demodulated signal*, \( y_b(t) \) is usually filtered with a filter \( g_R(t) \). Thereby, \( g_R(t) \) can be e.g. a lowpass filter or the optimum matched filter. The filtered received signal is
  \[ r_b(t) = g_R(t) * y_b(t) \]
  \[ = \sum_{k=-\infty}^{\infty} I[k] x(t - kT) + \nu(t) \]
where ”∗” denotes convolution, and we use the definitions
\[ x(t) = g_T(t) * g_R(t) = \int_{-\infty}^{\infty} g_T(\tau) g_R(t - \tau) \, d\tau \]
and
\[ \nu(t) = g_R(t) * z(t) \]

Now, we sample \( r_b(t) \) at times \( t = kT + t_0, k = \ldots, -1, 0, 1, \ldots \), where \( t_0 \) is an arbitrary delay. For simplicity, we assume \( t_0 = 0 \) in the following and obtain
\[ r_b[k] = r_b(kT) = \sum_{\kappa=-\infty}^{\infty} I[\kappa] x[kT - \kappa T] + \nu(kT) = \sum_{\kappa=-\infty}^{\infty} I[\kappa] x[k - \kappa] + z[k] \]
We can rewrite $r_b[k]$ as

$$r_b[k] = x[0] \left( I[k] + \frac{1}{x[0]} \sum_{\kappa=-\infty}^{\infty} I[\kappa] x[k - \kappa] \right) + z[k],$$

where the term

$$\sum_{\kappa=-\infty}^{\infty} I[\kappa] x[k - \kappa]$$

represents so-called *intersymbol interference (ISI)*. Since $x[0]$ only depends on the amplitude of $g_R(t)$ and $g_T(t)$, respectively, for simplicity we assume $x[0] = 1$ in the following.

Ideally, we would like to have

$$r_b[k] = I[k] + z[k],$$

which implies

$$\sum_{\kappa=-\infty}^{\infty} I[\kappa] x[k - \kappa] = 0,$$

i.e., the transmission is ISI free.

**Problem Statement**

How do we design $g_T(t)$ and $g_R(t)$ to guarantee ISI–free transmission?
**Solution: The Nyquist Criterion**

Since

\[ x(t) = g_R(t) \ast g_T(t) \]

is valid, the Fourier transform of \( x(t) \) can be expressed as

\[ X(f) = G_T(f) G_R(f), \]

with \( G_T(f) = \mathcal{F}\{g_T(t)\} \) and \( G_R(f) = \mathcal{F}\{g_R(t)\} \).

For ISI–free transmission, \( x(t) \) has to have the property

\[ x(kT) = x[k] = \begin{cases} 1, & k = 0 \\ 0, & k \neq 0 \end{cases} \quad (1) \]

**The Nyquist Criterion**

According to the Nyquist Criterion, condition (1) is fulfilled if and only if

\[ \sum_{m=-\infty}^{\infty} X\left(f + \frac{m}{T}\right) = T \]

is true. This means summing up all spectra that can be obtained by shifting \( X(f) \) by multiples of \( 1/T \) results in a constant.
Proof:  

\( x(t) \) may be expressed as

\[
x(t) = \int_{-\infty}^{\infty} X(f) e^{j2\pi ft} \, df.
\]

Therefore, \( x[k] = x(kT) \) can be written as

\[
x[k] = \int_{-\infty}^{\infty} X(f) e^{j2\pi f kT} \, df
\]

\[
= \sum_{m=-\infty}^{\infty} \int_{(2m-1)/(2T)}^{(2m+1)/(2T)} X(f) e^{j2\pi f kT} \, df
\]

\[
= \sum_{m=-\infty}^{\infty} \int_{-1/(2T)}^{1/(2T)} X(f' + m/T) e^{j2\pi (f' + m/T) kT} \, df'
\]

\[
= \int_{-1/(2T)}^{1/(2T)} \sum_{m=-\infty}^{\infty} X(f' + m/T) e^{j2\pi f' kT} \, df'
\]

\[
= \int_{-1/(2T)}^{1/(2T)} B(f') e^{j2\pi f kT} \, df'
\]

Since

\[
B(f) = \sum_{m=-\infty}^{\infty} X(f + m/T)
\]
is periodic with period $1/T$, it can be expressed as a **Fourier series**

$$B(f) = \sum_{k=-\infty}^{\infty} b[k] e^{j2\pi f kT}$$  \hspace{1cm} (3)

where the Fourier series coefficients $b[k]$ are given by

$$b[k] = T \int_{-1/(2T)}^{1/(2T)} B(f) e^{-j2\pi f kT} \, df.$$  \hspace{1cm} (4)

From (2) and (4) we observe that

$$b[k] = T x(-kT).$$

Therefore, (1) holds if and only if

$$b[k] = \begin{cases} T, & k = 0 \\ 0, & k \neq 0 \end{cases}.$$  

However, in this case (3) yields

$$B(f) = T,$$

which means

$$\sum_{m=-\infty}^{\infty} X(f + m/T) = T.$$

This completes the proof.
The Nyquist criterion specifies the necessary and sufficient condition that has to be fulfilled by the spectrum \( X(f) \) of \( x(t) \) for ISI–free transmission. Pulse shapes \( x(t) \) that satisfy the Nyquist criterion are referred to as \textit{Nyquist pulses}. In the following, we discuss three different cases for the symbol duration \( T \). In particular, we consider \( T < 1/(2W) \), \( T = 1/(2W) \), and \( T > 1/(2W) \), respectively, where \( W \) is the bandwidth of the ideally bandlimited channel.

1. \( T < 1/(2W) \)
   If the symbol duration \( T \) is smaller than \( 1/(2W) \), we get obviously
   \[
   \frac{1}{2T} > W
   \]
   and
   \[
   B(f) = \sum_{m=-\infty}^{\infty} X(f + m/T)
   \]
   consists of non–overlapping pulses.

   We observe that in this case the condition \( B(f) = T \) cannot be fulfilled. Therefore, ISI–free transmission is \textit{not possible}. 

\[ 
\text{Schober: Signal Detection and Estimation} 
\]
2. $T = 1/(2W)$

If the symbol duration $T$ is equal to $1/(2W)$, we get

$$\frac{1}{2T} = W$$

and $B(f) = T$ is achieved for

$$X(f) = \begin{cases} T, & |f| \leq W \\ 0, & |f| > W \end{cases}.$$ 

This corresponds to

$$x(t) = \frac{\sin\left(\frac{\pi t}{T}\right)}{\pi \frac{t}{T}}.$$ 

$T = 1/(2W)$ is also called the Nyquist rate and is the fastest rate for which ISI–free transmission is possible.

In practice, however, this choice is usually not preferred since $x(t)$ decays very slowly ($\propto 1/t$) and therefore, time synchronization is problematic.
3. $T > 1/(2W)$

In this case, we have

$$\frac{1}{2T} < W$$

and $B(f)$ consists of overlapping, shifted replica of $X(f)$. ISI-free transmission is possible if $X(f)$ is chosen properly.

Example:
The bandwidth occupied beyond the Nyquist bandwidth $1/(2T)$ is referred to as the **excess bandwidth**. In practice, Nyquist pulses with **raised-cosine** spectra are popular

$$X(f) = \begin{cases} T, & 0 \leq |f| \leq \frac{1-\beta}{2T} \\ \frac{T}{2} \left(1 + \cos \left( \frac{\pi T}{\beta} \left[ |f| - \frac{1-\beta}{2T} \right] \right) \right), & \frac{1-\beta}{2T} < |f| \leq \frac{1+\beta}{2T} \\ 0, & |f| > \frac{1+\beta}{2T} \end{cases}$$

where $\beta$, $0 \leq \beta \leq 1$, is the roll-off factor. The inverse Fourier transform of $X(f)$ yields

$$x(t) = \frac{\sin(\pi t/T) \cos(\pi \beta t/T)}{\frac{\pi t}{T}} \frac{1}{1 - 4\beta^2 t^2/T^2}.$$  

For $\beta = 0$, $x(t)$ reduces to $x(t) = \sin(\pi t/T)/(\pi t/T)$. For $\beta > 0$, $x(t)$ decays as $1/t^3$. This fast decay is desirable for time synchronization.
$x(t)$

$t/T$

$\beta = 0.35$

$\beta = 1$

$\beta = 0$

Schober: Signal Detection and Estimation
\[ \sqrt{\text{Nyquist}} - \text{Filters} \]

The spectrum of \( x(t) \) is given by

\[ G_T(f) G_R(f) = X(f) \]

ISI–free transmission is of course possible for any choice of \( G_T(f) \) and \( G_R(f) \) as long as \( X(f) \) fulfills the Nyquist criterion. However, the SNR maximizing optimum choice is

\[
\begin{align*}
G_T(f) &= G(f) \\
G_R(f) &= \alpha G^*(f),
\end{align*}
\]

i.e., \( g_R(t) \) is matched to \( g_T(t) \). \( \alpha \) is a constant. In that case, \( X(f) \) is given by

\[ X(f) = \alpha |G(f)|^2 \]

and consequently, \( G(f) \) can be expressed as

\[ G(f) = \sqrt{\frac{1}{\alpha} X(f)}. \]

\( G(f) \) is referred to as \( \sqrt{\text{Nyquist}} - \text{Filter} \). In practice, a delay \( \tau_0 \) may be added to \( G_T(f) \) and/or \( G_R(f) \) to make them causal filters in order to ensure physical realizability of the filters.
5.3 Discrete–Time Channel Model for ISI–free Transmission

- Continuous–Time Channel Model

\[ I[k] \xrightarrow{g_T(t)} c(t) \xrightarrow{g_R(t)} z(t) \xrightarrow{r_b(t) kT} r_b[k] \]

\( g_T(t) \) and \( g_R(t) \) are \( \sqrt{\text{Nyquist}} \)-Impulses that are matched to each other.

- Equivalent Discrete–Time Channel Model

\[ I[k] \xrightarrow{\sqrt{E_g}} z[k] \xrightarrow{r_b[k]} \]

\( z[k] \) is a discrete–time AWGN process with variance \( \sigma_Z^2 = N_0 \).

**Proof:**

- *Signal Component*

  The overall channel is given by

  \[ x[k] = g_T(t) * c(t) * g_R(t) \bigg|_{t=kT} = g_T(t) * g_R(t) \bigg|_{t=kT} \]
where we have used the fact that the channel acts as an ideal lowpass filter. We assume

\[ g_R(t) = g(t) \]
\[ g_T(t) = \frac{1}{\sqrt{E_g}} g^*(-t), \]

where \( g(t) \) is a \( \sqrt{\text{Nyquist}} \)-Impulse, and \( E_g \) is given by

\[ E_g = \int_{-\infty}^{\infty} |g(t)|^2 \, dt \]

Consequently, \( x[k] \) is given by

\[
x[k] = \frac{1}{\sqrt{E_g}} g(t) * g^*(-t) \bigg|_{t=kT} \\
= \frac{1}{\sqrt{E_g}} g(t) * g^*(-t) \bigg|_{t=0} \delta[k] \\
= \frac{1}{\sqrt{E_g}} E_g \delta[k] \\
= \sqrt{E_g} \delta[k]
\]

- \textit{Noise Component}
  
  \( z(t) \) is a complex AWGN process with power spectral density
\[ \Phi_{ZZ}(f) = N_0. \] 
\[ z[k] \text{ is given by} \]
\[ z[k] = g_R(t) * z(t) \bigg|_{t=kT} \]
\[ = \frac{1}{\sqrt{E_g}} g^*(-t) * z(t) \bigg|_{t=kT} \]
\[ = \frac{1}{\sqrt{E_g}} \int_{-\infty}^{\infty} z(\tau) g^*(kT + \tau) \, d\tau \]

*Mean*

\[ \mathcal{E}\{z[k]\} = \mathcal{E} \left\{ \frac{1}{\sqrt{E_g}} \int_{-\infty}^{\infty} z(\tau) g^*(kT + \tau) \, d\tau \right\} \]
\[ = \frac{1}{\sqrt{E_g}} \int_{-\infty}^{\infty} \mathcal{E}\{z(\tau)\} g^*(kT + \tau) \, d\tau \]
\[ = 0 \]
* ACF

\[ \phi_{ZZ}[\lambda] = \mathcal{E}\{z[k + \lambda]z^*[k]\} \]

\[ = \frac{1}{E_g} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \mathcal{E}\{z(\tau_1)z^*(\tau_2)\} g^*([k + \lambda]T + \tau_1) \cdot g(kT + \tau_2) \, d\tau_1 \, d\tau_2 \]

\[ = \frac{N_0}{E_g} \int_{-\infty}^{\infty} g^*([k + \lambda]T + \tau_1) g(kT + \tau_1) \, d\tau_1 \]

\[ = \frac{N_0}{E_g} \delta[\lambda] \]

\[ = N_0 \delta[\lambda] \]

This shows that \( z[k] \) is a discrete–time AWGN process with variance \( \sigma_Z^2 = N_0 \) and the proof is complete.

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- The discrete–time, demodulated received signal can be modeled as

\[ r_b[k] = \sqrt{E_g} I[k] + z[k]. \]

- This means the demodulated signal is identical to the demodulated signal obtained for time–limited transmit pulses (see Chapter 3 and 4). Therefore, the optimum detection strategies developed in Chapter 4 can be also applied for finite bandwidth transmission as long as the Nyquist criterion is fulfilled.