6 Equalization of Channels with ISI

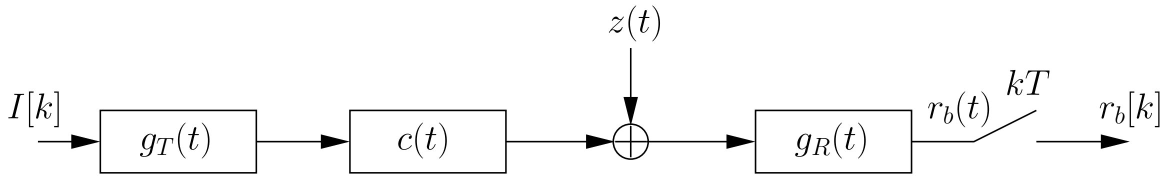
- Many practical channels are bandlimited and linearly distort the transmit signal.
- In this case, the resulting ISI channel has to be equalized for reliable detection.
- There are many different equalization techniques. In this chapter, we will discuss the three most important equalization schemes:
  1. Maximum–Likelihood Sequence Estimation (MLSE)
  2. Linear Equalization (LE)
  3. Decision–Feedback Equalization (DFE)

  Throughout this chapter we assume linear memoryless modulations such as PAM, PSK, and QAM.

6.1 Discrete–Time Channel Model

■ Continuous–Time Channel Model

The continuous–time channel is modeled as shown below.



– *Channel* $c(t)$
In general, the channel $c(t)$ is not ideal, i.e., $|C(f)|$ is not a constant over the range of frequencies where $G_T(f)$ is non–zero. Therefore, linear distortions are inevitable.

– *Transmit Filter* $g_T(t)$
The transmit filter $g_T(t)$ may or may not be a $\sqrt{\text{Nyquist}}$–Filter, e.g. in the North American D–AMPS mobile phone system a square–root raised cosine filter with roll–off factor $\beta = 0.35$ is used, whereas in the European EDGE mobile communication system a linearized Gaussian minimum–shift keying (GMSK) pulse is employed.

– *Receive Filter* $g_R(t)$
We assume that the receive filter $g_R(t)$ is a $\sqrt{\text{Nyquist}}$–Filter. Therefore, the filtered, sampled noise $z[k] = g_R(t) * z(t)|_{t=kT}$ is white Gaussian noise (WGN).

Ideally, $g_R(t)$ consists of a filter matched to $g_T(t) * c(t)$ and a noise whitening filter. The drawback of this approach is that $g_R(t)$ depends on the channel, which may change with time in wireless applications. Therefore, in practice often a fixed but suboptimum $\sqrt{\text{Nyquist}}$–Filter is preferred.

– *Overall Channel* $h(t)$

The overall channel impulse response $h(t)$ is given by

$$h(t) = g_T(t) * c(t) * g_R(t).$$

- **Discrete–Time Channel Model**

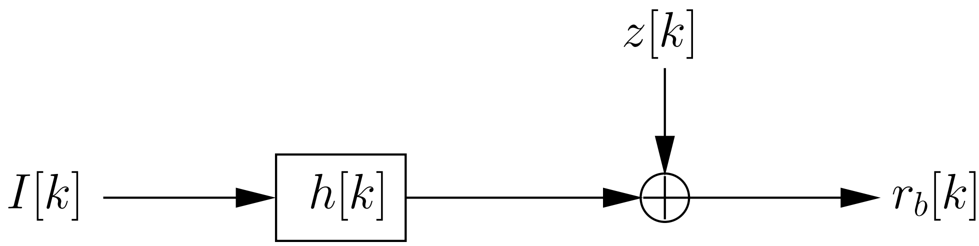
The sampled received signal is given by

$$\begin{aligned} r_b[k] &= r_b(kT) \\ &= \left.\left(\sum_{m=-\infty}^{\infty} I[m]h(t-mT) + g_R(t) * z(t)\right)\right|_{t=kT} \\ &= \sum_{m=-\infty}^{\infty} I[m]\underbrace{h(kT-mT)}_{=h[k-m]} + \underbrace{g_R(t) * z(t)\Big|_{t=kT}}_{=z[k]} \\ &= \sum_{l=-\infty}^{\infty} h[l]I[k-l] + z[k], \end{aligned}$$

where $z[k]$ is AWGN with variance $\sigma_Z^2 = N_0$, since $g_R(t)$ is a $\sqrt{\text{Nyquist}}$–Filter.

In practice, $h[l]$ can be truncated to some finite length L. If we assume causality of $g_T(t)$, $g_R(t)$, and $c(t)$, $h[l] = 0$ holds for $l < 0$, and if L is chosen large enough $h[l] \approx 0$ holds also for $l \geq L$. Therefore, $r_b[k]$ can be rewritten as

$$\boxed{r_b[k] = \sum_{l=0}^{L-1} h[l]I[k-l] + z[k]}$$



For all equalization schemes derived in the following, it is assumed that the overall channel impulse response $h[k]$ is perfectly known, and only the transmitted information symbols $I[k]$ have to be estimated. In practice, $h[k]$ is unknown, of course, and has to be estimated first. However, this is not a major problem and can be done e.g. using a training sequence of known symbols.

6.2 Maximum–Likelihood Sequence Estimation (MLSE)

- We consider the transmission of a block of K unknown information symbols $I[k]$, $0 \leq k \leq K-1$, and assume that $I[k]$ is known for $k < 0$ and $k \geq K$, respectively.
- We collect the transmitted information sequence $\{I[k]\}$ in a vector

$$\boldsymbol{I} = [I[0] \;\ldots\; I[K-1]]^T$$

and the corresponding vector of discrete–time received signals is given by

$$\boldsymbol{r}_b = [r_b[0] \;\ldots\; r_b[K+L-2]]^T.$$

Note that $r_b[K+L-2]$ is the last received signal that contains $I[K-1]$.

- **ML Detection**

  For ML detection, we need the pdf $p(\boldsymbol{r}_b|\boldsymbol{I})$ which is given by

$$p(\boldsymbol{r}_b|\boldsymbol{I}) \propto \exp\left(-\frac{1}{N_0}\sum_{k=0}^{K+L-2}\left|r_b[k]-\sum_{l=0}^{L-1}h[l]I[k-l]\right|^2\right).$$

  Consequently, the ML detection rule is given by

$$\begin{aligned}\hat{\boldsymbol{I}} &= \underset{\tilde{\boldsymbol{I}}}{\operatorname{argmax}}\left\{p(\boldsymbol{r}_b|\tilde{\boldsymbol{I}})\right\}\\ &= \underset{\tilde{\boldsymbol{I}}}{\operatorname{argmax}}\left\{\ln[p(\boldsymbol{r}_b|\tilde{\boldsymbol{I}})]\right\}\\ &= \underset{\tilde{\boldsymbol{I}}}{\operatorname{argmin}}\left\{-\ln[p(\boldsymbol{r}_b|\tilde{\boldsymbol{I}})]\right\}\\ &= \underset{\tilde{\boldsymbol{I}}}{\operatorname{argmin}}\left\{\sum_{k=0}^{K+L-2}\left|r_b[k]-\sum_{l=0}^{L-1}h[l]\tilde{I}[k-l]\right|^2\right\},\end{aligned}$$

  where $\hat{\boldsymbol{I}}$ and $\tilde{\boldsymbol{I}}$ denote the estimated sequence and a trial sequence, respectively. Since the above decision rule suggest that we detect the entire sequence $\boldsymbol{I}$ based on the received sequence $\boldsymbol{r}_b$, this optimal scheme is known as *Maximum–Likelihood Sequence Estimation (MLSE)*.

- Notice that there are M^K different trial sequences/vectors $\tilde{\boldsymbol{I}}$ if M–ary modulation is used. Therefore, the complexity of MLSE with brute–force search is exponential in the sequence length K. This is not acceptable for a practical implementation even for relatively small sequence lengths. Fortunately, the exponential complexity in K can be overcome by application of the *Viterbi Algorithm (VA)*.

- **Viterbi Algorithm (VA)**

For application of the VA we need to define a *metric* that can be computed, recursively. Introducing the definition

$$\Lambda[k+1] = \sum_{m=0}^{k} \left| r_b[m] - \sum_{l=0}^{L-1} h[l]\tilde{I}[m-l] \right|^2,$$

we note that the function to be minimized for MLSE is $\Lambda[K+L-1]$. On the other hand,

$$\Lambda[k+1] = \Lambda[k] + \lambda[k]$$

with

$$\lambda[k] = \left| r_b[k] - \sum_{l=0}^{L-1} h[l]\tilde{I}[k-l] \right|^2$$

is valid, i.e., $\Lambda[k+1]$ can be calculated recursively from $\Lambda[k]$, which renders the application of the VA possible.

For M–ary modulation an ISI channel of length L can be described by a trellis diagram with M^{L-1} states since the signal component

$$\sum_{l=0}^{L-1} h[l]\tilde{I}[k-l]$$

can assume M^L different values that are determined by the M^{L-1} states

$$\boldsymbol{S}[k] = [\tilde{I}[k-1], \ldots, \tilde{I}[k-(L-1)]]$$

and the M possible transitions $\tilde{I}[k]$ to state

$$\boldsymbol{S}[k+1] = [\tilde{I}[k], \ldots, \tilde{I}[k-(L-2)]].$$

Therefore, the VA operates on a trellis with M^{L-1} states.

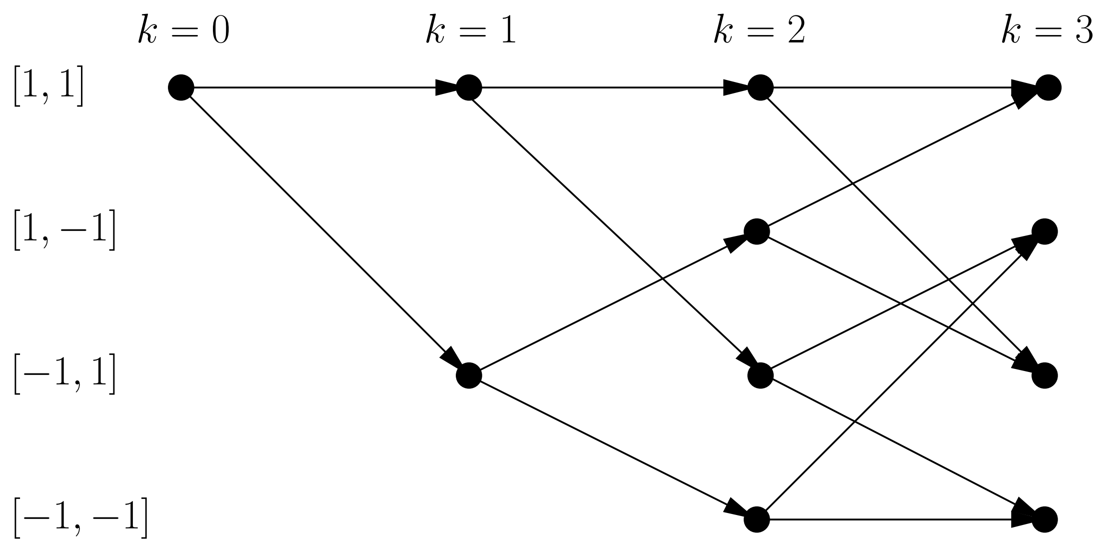
Example:

We explain the VA more in detail using an example. We assume BPSK transmission, i.e., $I[k] \in \{\pm 1\}$, and $L = 3$. For $k < 0$ and $k \geq K$, we assume that $I[k] = 1$ is transmitted.

– There are $M^{L-1} = 2^2 = 4$ states, and $M = 2$ transitions per state. State $\boldsymbol{S}[k]$ is defined as

$$\boldsymbol{S}[k] = [\tilde{I}[k-1],\ \tilde{I}[k-2]]$$

– Since we know that $I[k] = 1$ for $k < 0$, state $\boldsymbol{S}[0] = [1,\ 1]$ holds, whereas $\boldsymbol{S}[1] = [\tilde{I}[0],\ 1]$, and $\boldsymbol{S}[2] = [\tilde{I}[1],\ \tilde{I}[0]]$, and so on. The resulting trellis is shown below.



– $k = 0$
Arbitrarily and without loss of optimality, we may set the *accumulated* metric corresponding to state $\boldsymbol{S}[k]$ at time $k = 0$ equal to zero

$$\Lambda(\boldsymbol{S}[0], 0) = \Lambda([1,\ 1], 0) = 0.$$

Note that there is only one accumulated metric at time $k = 0$ since $\boldsymbol{S}[0]$ is known at the receiver.

– $k = 1$
The accumulated metric corresponding to $\boldsymbol{S}[1] = [\tilde{I}[0],\ 1]$ is given by

$$\begin{aligned}\Lambda(\boldsymbol{S}[1], 1) &= \Lambda(\boldsymbol{S}[0], 0) + \lambda(\boldsymbol{S}[0], \tilde{I}[0], 0) \\ &= \lambda(\boldsymbol{S}[0], \tilde{I}[0], 0)\end{aligned}$$

Since there are two possible states, namely $\boldsymbol{S}[1] = [1,\ 1]$ and $\boldsymbol{S}[1] = [-1,\ 1]$, there are two corresponding accumulated metrics at time $k = 1$.

– $k = 2$
Now, there are 4 possible states $\boldsymbol{S}[2] = [\tilde{I}[1],\ \tilde{I}[0]]$ and for each state a corresponding accumulated metric

$$\Lambda(\boldsymbol{S}[2], 2) = \Lambda(\boldsymbol{S}[1], 1) + \lambda(\boldsymbol{S}[1], \tilde{I}[1], 1)$$

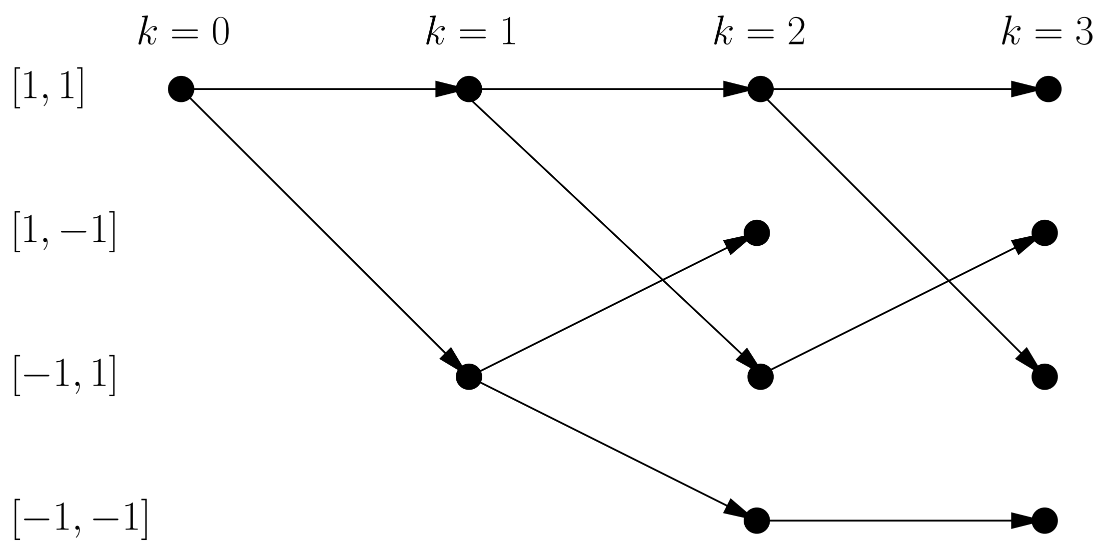
has to be calculated.

– $k = 3$

At $k = 3$ two branches emanate in each state $\boldsymbol{S}[3]$. However, since of the two paths that emanate in the same state $\boldsymbol{S}[3]$ that path which has the smaller accumulated metric $\Lambda(\boldsymbol{S}[3], 3)$ also will have the smaller metric at time $k = K+L-2 = K+1$, we need to retain only the path with the smaller $\Lambda(\boldsymbol{S}[3], 3)$. This path is also referred to as the *surviving path*. In mathematical terms, the accumulated metric for state $\boldsymbol{S}[3]$ is given by

$$\Lambda(\boldsymbol{S}[3], 3) = \underset{\tilde{I}[2]}{\operatorname{argmin}} \{\Lambda(\boldsymbol{S}[2], 2) + \lambda(\boldsymbol{S}[2], \tilde{I}[2], 2)\}$$

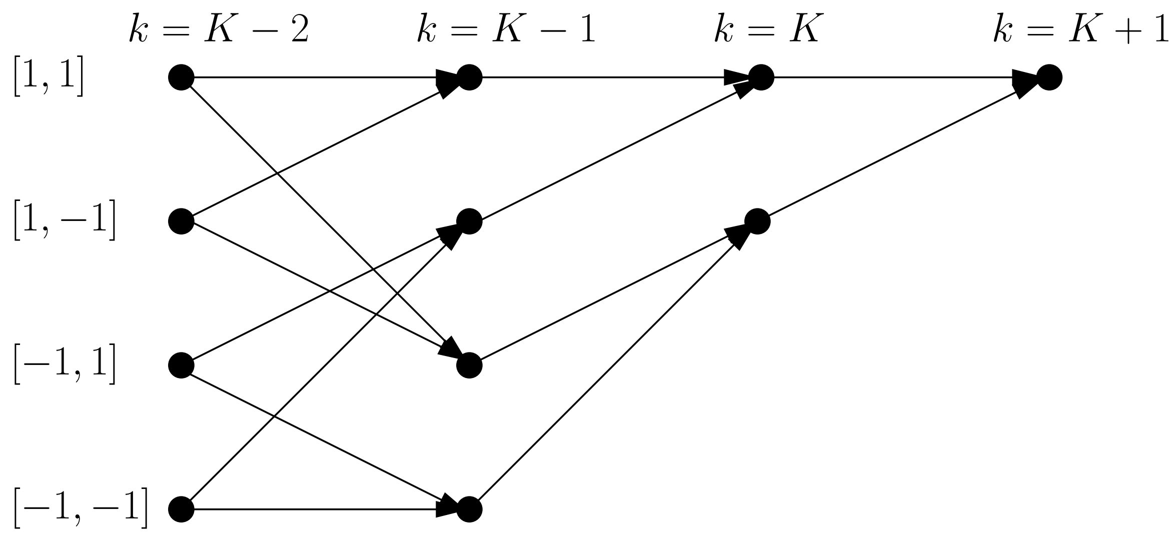
If we retain only the surviving paths, the above trellis at time $k = 3$ may be as shown be below.



– $k \geq 4$

All following steps are similar to that at time $k = 3$. In each step k we retain only $M^{L-1} = 4$ surviving paths and the corresponding accumulated branch metrics.

– *Termination of Trellis*
Since we assume that for $k \geq K$, $I[k] = 1$ is transmitted, the end part of the trellis is as shown below.



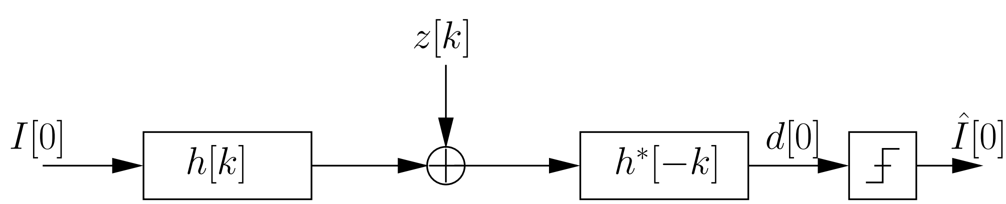
At time $k = K + L - 2 = K + 1$, there is only one surviving path corresponding to the ML sequence.

- Since only M^{L-1} paths are retained at each step of the VA, the complexity of the VA is *linear* in the sequence length K, but exponential in the length L of the overall channel impulse response.
- If the VA is implemented as described above, a decision can be made only at time $k = K+L-2$. However, the related delay may be unacceptable for large sequence lengths K. Fortunately, empirical studies have shown that the surviving paths tend to *merge* relatively quickly, i.e., at time k a decision can be made on the symbol $I[k - k_0]$ if the delay k_0 is chosen large enough. In practice, $k_0 \approx 5(L-1)$ works well and gives almost optimum results.

- **Disadvantage of MLSE with VA**
  In practice, the complexity of MLSE using the VA is often still too high. This is especially true if M is larger than 2. For those cases other, suboptimum equalization strategies have to be used.

- *Historical Note*
  MLSE using the VA in the above form has been introduced by Forney in 1972. Another variation was given later by Ungerböck in 1974. Ungerböck's version uses a matched filter at the receiver but does not require noise whitening.

- *Lower Bound on Performance*
  Exact calculation of the SEP or BEP of MLSE is quite involved and complicated. However, a simple lower bound on the performance of MLSE can be obtained by assuming that just one symbol $I[0]$ is transmitted. In that way, possibly detrimental interference from neighboring symbols is avoided. It can be shown that the optimum ML receiver for that scenario includes a filter matched to $h[k]$ and a decision can be made only based on the matched filter output at time $k = 0$.



The decision variable $d[0]$ is given by

$$d[0] = \sum_{l=0}^{L-1} |h[l]|^2 \, I[0] + \sum_{l=0}^{L-1} h^*[-l] z[-l].$$

We can model $d[0]$ as

$$d[0] = E_h I[0] + z_0[0],$$

where

$$E_h = \sum_{l=0}^{L-1} |h[l]|^2$$

and $z_0[0]$ is Gaussian noise with variance

$$\begin{aligned}\sigma_0^2 &= \mathcal{E}\left\{\left|\sum_{l=0}^{L-1} h^*[-l]z[-l]\right|^2\right\} \\ &= E_h\sigma_Z^2 = E_h N_0\end{aligned}$$

Therefore, this corresponds to the transmission of $I[0]$ over a *non–ISI* channel with E_S/N_0 ratio

$$\frac{E_S}{N_0} = \frac{E_h^2}{E_h N_0} = \frac{E_h}{N_0},$$

and the related SEP or BEP can be calculated easily. For example, for the BEP of BPSK we obtain

$$P_{\mathrm{MF}} = Q\left(\sqrt{2\,\frac{E_h}{N_0}}\right).$$

For the true BEP of MLSE we get

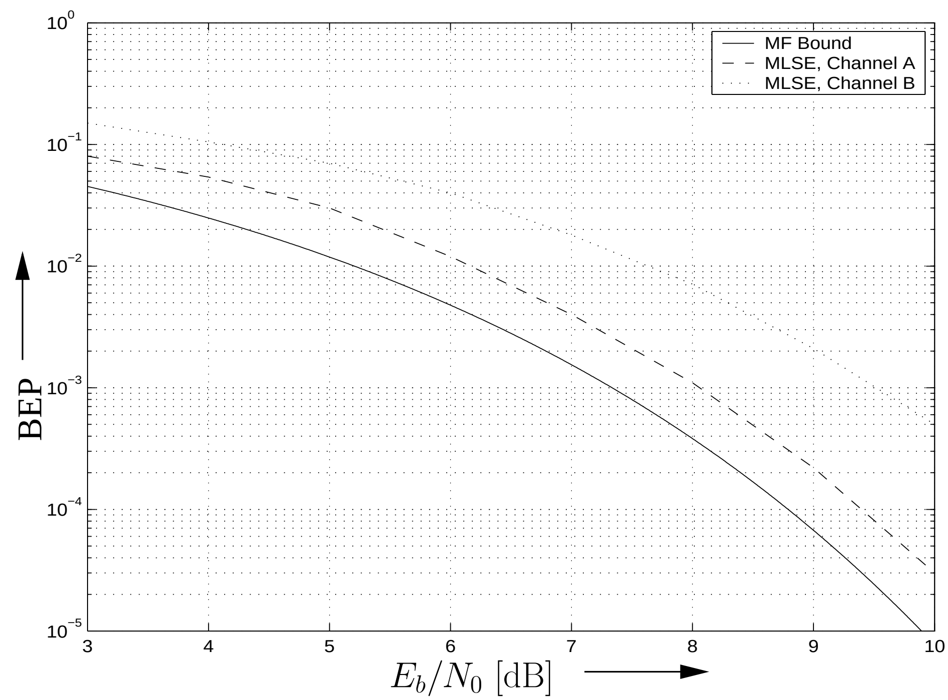
$$P_{\mathrm{MLSE}} \geq P_{\mathrm{MF}}.$$

The above bound is referred to as the *matched–filter (MF)* bound. The tightness of the MF bound largely depends on the underlying channel. For example, for a channel with $L = 2$, $h[0] = h[1] = 1$

and BPSK modulation the loss of MLSE compared to the MF bound is 3 dB. On the other hand, for random channels as typically encountered in wireless communications the MF bound is relatively tight.

Example:

For the following example we define two test channels of length $L = 3$. Channel A has an impulse response of $h[0] = 0.304$, $h[1] = 0.903$, $h[2] = 0.304$, whereas the impulse response of Channel B is given by $h[0] = 1/\sqrt{6}$, $h[1] = 2/\sqrt{6}$, $h[2] = 1/\sqrt{6}$. The received energy per symbol is in both cases $E_S = E_h = 1$. Assuming QPSK transmission, the received energy per bit E_b is $E_b = E_S/2$. The performance of MLSE along with the corresponding MF bound is shown below.

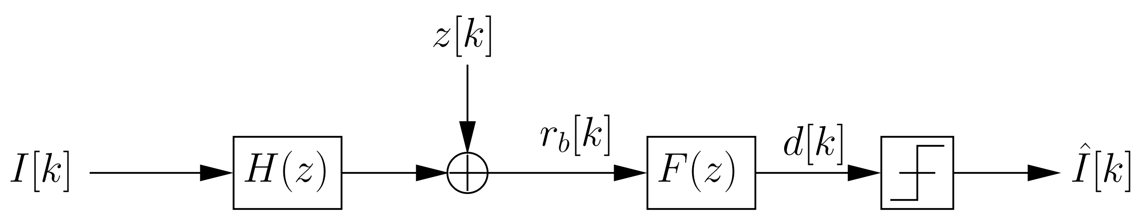


6.3 Linear Equalization (LE)

- Since MLSE becomes too complex for long channel impulse responses, in practice, often suboptimum equalizers with a lower complexity are preferred.
- The most simple suboptimum equalizer is the so–called *linear equalizer*. Roughly speaking, in LE a linear filter

$$F(z) = Z\{f[k]\} = \sum_{k=-\infty}^{\infty} f[k]z^{-k}$$

is used to *invert* the channel transfer function $H(z) = Z\{h[k]\}$, and symbol–by–symbol decisions are made subsequently. $f[k]$ denotes the equalizer filter coefficients.



Linear equalizers are categorized with respect to the following two criteria:

1. Optimization criterion used for calculation of the filter coefficients $f[k]$. Here, we will adopt the so–called *zero–forcing (ZF)* criterion and the *minimum mean–squared error (MMSE)* criterion.
2. Finite length vs. infinite length equalization filters.

6.3.1 Optimum Linear Zero–Forcing (ZF) Equalization

- Optimum ZF equalization implies that we allow for equalizer filters with infinite length impulse response (IIR).
- *Zero–forcing* means that it is our aim to force the residual intersymbol interference in the decision variable $d[k]$ to zero.
- Since we allow for IIR equalizer filters $F(z)$, the above goal can be achieved by

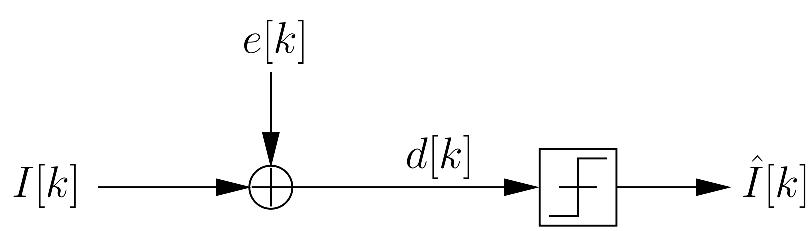
$$\boxed{F(z) = \frac{1}{H(z)}}$$

  where we assume that $H(z)$ has no roots on the unit circle. Since in most practical applications $H(z)$ can be modeled as a filter with finite impulse response (FIR), $F(z)$ will be an IIR filter in general.

- Obviously, the resulting overall channel transfer function is

$$H_{\text{ov}}(z) = H(z)F(z) = 1,$$

  and we arrive at the equivalent channel model shown below.



- The decision variable $d[k]$ is given by

$$d[k] = I[k] + e[k]$$

where $e[k]$ is *colored Gaussian noise* with power spectral density

$$\begin{aligned}\Phi_{ee}(e^{j2\pi fT}) &= N_0\,|F(e^{j2\pi fT})|^2 \\ &= \frac{N_0}{|H(e^{j2\pi fT})|^2}.\end{aligned}$$

The corresponding error variance can be calculated to

$$\begin{aligned}\sigma_e^2 &= \mathcal{E}\{|e[k]|^2\} \\ &= T \int\limits_{-1/(2T)}^{1/(2T)} \Phi_{ee}(e^{j2\pi fT})\,\mathrm{d}f \\ &= T \int\limits_{-1/(2T)}^{1/(2T)} \frac{N_0}{|H(e^{j2\pi fT})|^2}\,\mathrm{d}f.\end{aligned}$$

The signal–to–noise ratio (SNR) is given by

$$\boxed{\mathrm{SNR}_{\mathrm{IIR-ZF}} = \frac{\mathcal{E}\{|I[k]|^2\}}{\sigma_e^2} = \frac{1}{T \int\limits_{-1/(2T)}^{1/(2T)} \frac{N_0}{|H(e^{j2\pi fT})|^2}\,\mathrm{d}f}}$$

- We may consider two extreme cases for $H(z)$:
  1. $|H(e^{j2\pi fT})| = \sqrt{E_h}$
     If $H(z)$ has an allpass characteristic $|H(e^{j2\pi fT})| = \sqrt{E_h}$, we get $\sigma_e^2 = N_0/E_h$ and
     $$\mathrm{SNR}_{\mathrm{IIR-ZF}} = \frac{E_h}{N_0}.$$
     This is the same SNR as for an undistorted AWGN channel, i.e., no performance loss is suffered.
  2. *$H(z)$ has zeros close to the unit circle.*
     In that case $\sigma_e^2 \to \infty$ holds and
     $$\mathrm{SNR}_{\mathrm{IIR-ZF}} \to 0$$
     follows. In this case, ZF equalization leads to a very poor performance. Unfortunately, for wireless channels the probability of zeros close to the unit circle is very high. Therefore, linear ZF equalizers are not employed in wireless receivers.

- **Error Performance**
  Since optimum ZF equalization results in an equivalent channel with additive Gaussian noise, the corresponding BEP and SEP can be easily computed. For example, for BPSK transmission we get
  $$\boxed{P_{\mathrm{IIR-ZF}} = Q\left(\sqrt{2\,\mathrm{SNR}_{\mathrm{IIR-ZF}}}\right)}$$

Example:

We consider a channel with two coefficients and energy 1

$$H(z) = \frac{1}{\sqrt{1+|c|^2}}\,(1 - cz^{-1}),$$

where c is complex. The equalizer filter is given by

$$F(z) = \sqrt{1+|c|^2}\,\frac{z}{z-c}$$

In the following, we consider two cases: $|c| < 1$ and $|c| > 1$.

1. $|c| < 1$
   In this case, a stable, *causal* impulse response is obtained.

$$f[k] = \sqrt{1+|c|^2}\,c^k u[k],$$

   where $u[k]$ denotes the unit step function. The corresponding error variance is

$$\begin{aligned}
\sigma_e^2 &= T \int\limits_{-1/(2T)}^{1/(2T)} \frac{N_0}{|H(e^{j2\pi fT})|^2}\,\mathrm{d}f \\
&= N_0 T \int\limits_{-1/(2T)}^{1/(2T)} |F(e^{j2\pi fT})|^2\,\mathrm{d}f \\
&= N_0 \sum_{k=-\infty}^{\infty} |f[k]|^2 \\
&= N_0(1+|c|^2) \sum_{k=0}^{\infty} |c|^{2k} \\
&= N_0\,\frac{1+|c|^2}{1-|c|^2}.
\end{aligned}$$

The SNR becomes

$$\mathrm{SNR}_{\mathrm{IIR-ZF}} = \frac{1}{N_0} \frac{1-|c|^2}{1+|c|^2}.$$

2. $|c| > 1$

Now, we can realize the filter as stable and *anti–causal* with impulse response

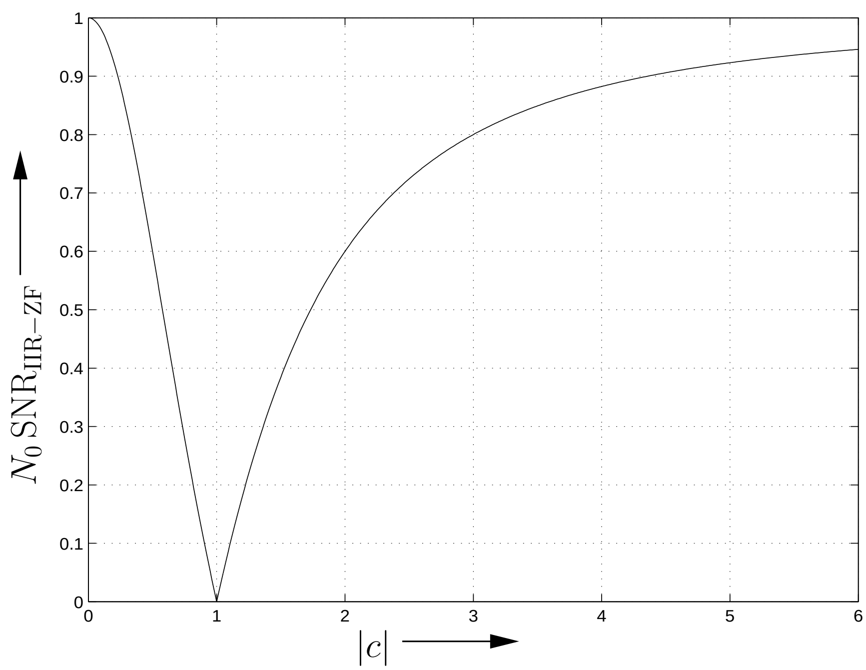
$$f[k] = -\frac{\sqrt{1+|c|^2}}{c} c^{k+1} u[-(k+1)].$$

Using similar techniques as above, the error variance becomes

$$\sigma_e^2 \;=\; N_0 \frac{1+|c|^2}{|c|^2-1},$$

and we get for the SNR

$$\mathrm{SNR}_{\mathrm{IIR-ZF}} = \frac{1}{N_0} \frac{|c|^2-1}{1+|c|^2}.$$



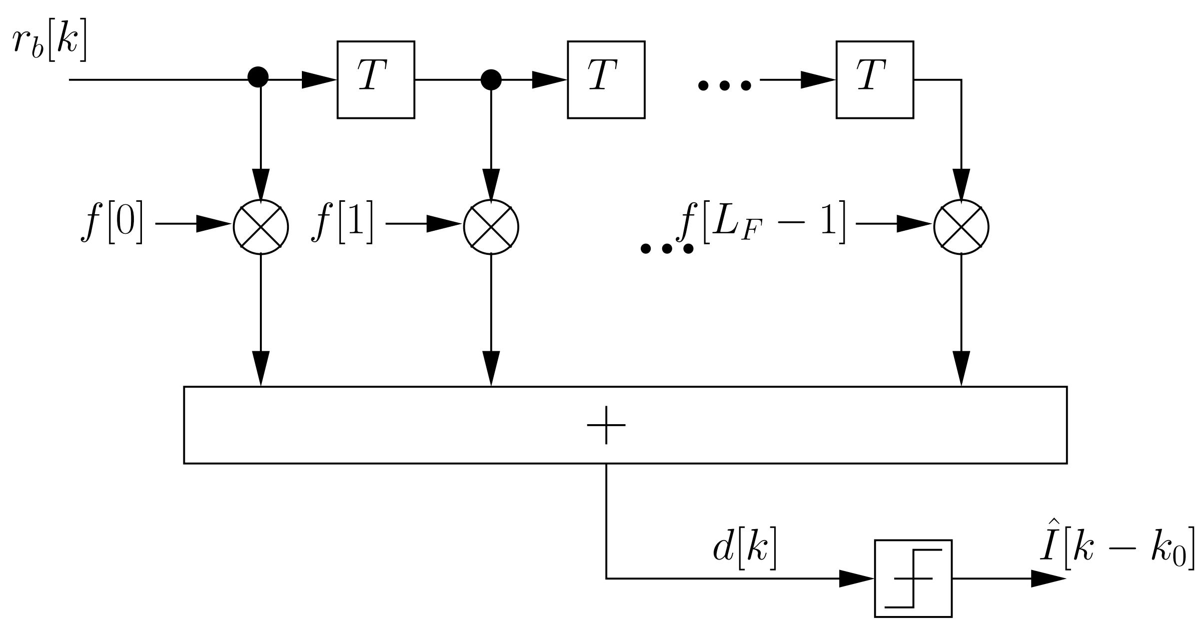
Obviously, the SNR drops to zero as $|c|$ approaches one, i.e., as the root of $H(z)$ approaches the unit circle.

6.3.2 ZF Equalization with FIR Filters

- In this case, we impose a causality and a length constraint on the equalizer filter and the transfer function is given by

$$F(z) = \sum_{k=0}^{L_F-1} f[k]z^{-k}$$

In order to be able to deal with "non–causal components", we introduce a decision delay $k_0 \geq 0$, i.e., at time k, we estimate $I[k-k_0]$. Here, we assume a fixed value for k_0, but in practice, k_0 can be used for optimization.



- Because of the finite filter length, a complete elimination of ISI is in general not possible.

- **Alternative Criterion**: Peak–Distortion Criterion

  Minimize the maximum possible distortion of the signal at the equalizer output due to ISI.

- **Optimization**

  In mathematical terms the above criterion can be formulated as follows.

  Minimize

$$D = \sum_{\substack{k=-\infty \\ k \neq k_0}}^{\infty} |h_{\text{ov}}[k]|$$

  subject to

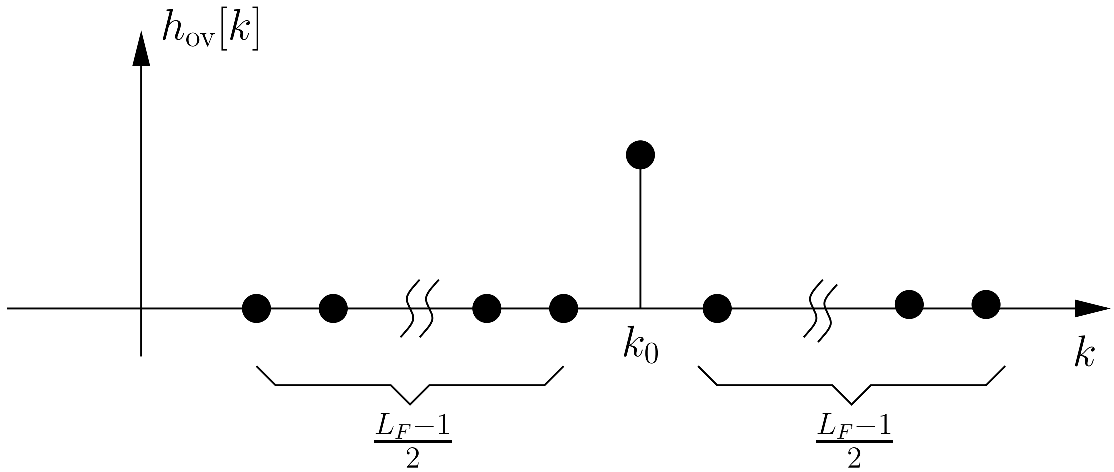
$$h_{\text{ov}}[k_0] = 1,$$

  where $h_{\text{ov}}[k]$ denotes the overall impulse response (channel and equalizer filter).

  Although D is a convex function of the equalizer coefficients, it is in general difficult to find the optimum filter coefficients. An exception is the special case when the *binary eye* at the equalizer input is open

$$\frac{1}{|h[k_1]|} \sum_{\substack{k=-\infty \\ k \neq k_1}}^{\infty} |h[k]| < 1$$

  for some k_1. In this case, if we assume furthermore $k_0 = k_1 + (L_F - 1)/2$ (L_F odd), D is minimized if and only if the overall impulse response $h_{\text{ov}}[k]$ has $(L_F - 1)/2$ consecutive zeros to the left and to the right of $h_{\text{ov}}[k_0] = 1$.



- This shows that in this special case the Peak–Distortion Criterion corresponds to the *ZF criterion* for equalizers with finite order. Note that there is no restriction imposed on the remaining coefficients of $h_{\mathrm{ov}}[k]$ ("don't care positions").

- **Problem**
  If the binary eye at the equalizer input is closed, in general, D is *not* minimized by the ZF solution. In this case, the coefficients at the "don't care positions" may take on large values.

- **Calculation of the ZF Solution**
  The above ZF criterion leads us to the conditions

$$h_{\mathrm{ov}}[k] = \sum_{m=0}^{q_F} f[m]h[k-m] = 0$$

where $k \in \{k_0 - q_F/2, \ldots, k_0 - 1, k_0 + 1, \ldots, k_0 + q_F/2\}$, and

$$h_{\mathrm{ov}}[k_0] = \sum_{m=0}^{q_F} f[m]h[k_0-m] = 1,$$

and $q_F = L_F - 1$. The resulting system of linear equations to be solved can be written as

$$\boldsymbol{H}\boldsymbol{f} = [0\ 0\ \ldots\ 0\ 1\ 0\ \ldots\ 0]^T,$$

with the $L_F \times L_F$ matrix

$$\boldsymbol{H} = \begin{bmatrix} h[k_0 - q_F/2] & h[k_0 - q_F/2 - 1] & \cdots & h[k_0 - 3q_F/2] \\ h[k_0 - q_F/2 + 1] & h[k_0 - q_F/2] & \cdots & h[k_0 - 3q_F/2 + 1] \\ \vdots & \vdots & \vdots & \vdots \\ h[k_0 - 1] & h[k_0 - 2] & \cdots & h[k_0 - q_F - 1] \\ h[k_0] & h[k_0 - 1] & \cdots & h[k_0 - q_F] \\ h[k_0 + 1] & h[k_0] & \cdots & h[k_0 - q_F + 1] \\ \vdots & \vdots & \vdots & \vdots \\ h[k_0 + q_F/2] & h[k_0 + q_F/2 - 1] & \cdots & h[k_0 - q_F/2] \end{bmatrix}$$

and coefficient vector

$$\boldsymbol{f} = [f[0]\ f[1]\ \ldots\ f[q_F]]^T.$$

The ZF solution is given by

$$\boxed{\boldsymbol{f} = \boldsymbol{H}^{-1}\,[0\ 0\ \ldots\ 0\ 1\ 0\ \ldots\ 0]^T}$$

or in other words, the optimum vector is the $(q_F/2 + 1)$th row of the inverse of $\boldsymbol{H}$.

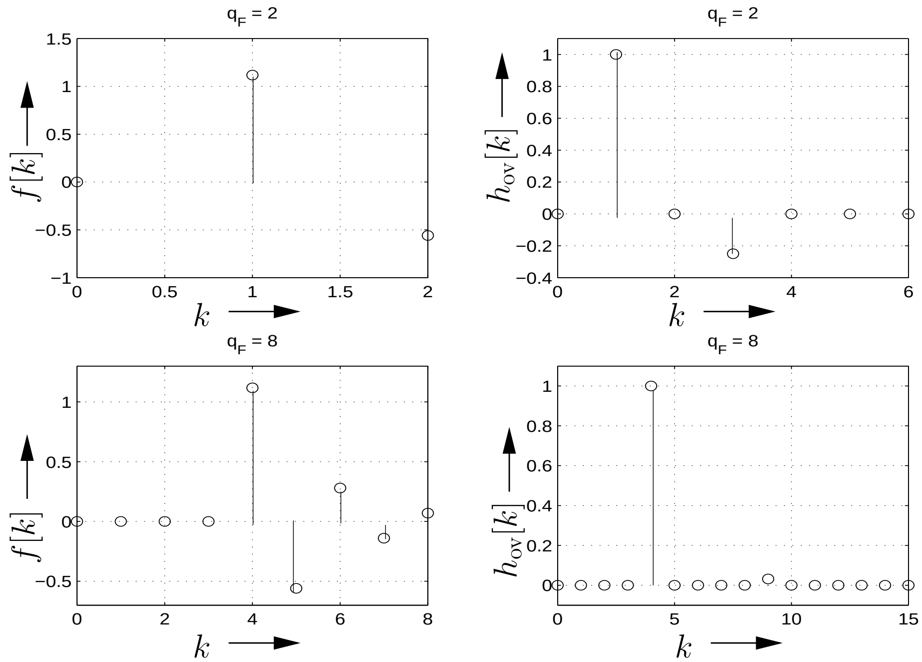
Example:

We assume

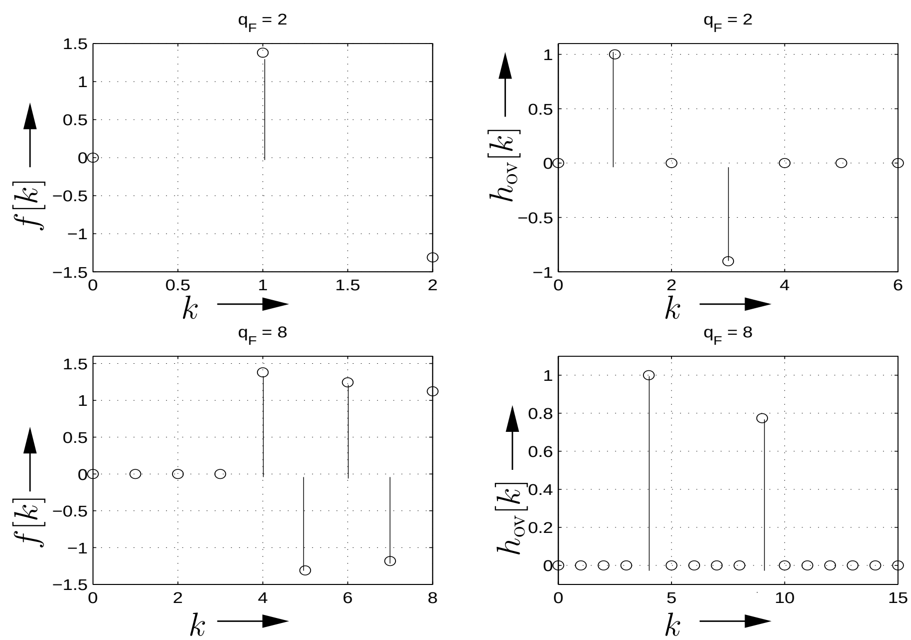
$$H(z) = \frac{1}{\sqrt{1 + |c|^2}}\,(1 - cz^{-1}),$$

and $k_0 = q_F/2$.

1. First we assume $c = 0.5$, i.e., $h[k]$ is given by $h[0] = 2/\sqrt{5}$ and $h[1] = 1/\sqrt{5}$.



2. In our second example we have $c = 0.95$, i.e., $h[k]$ is given by $h[0] = 0.73$ and $h[1] = 0.69$.



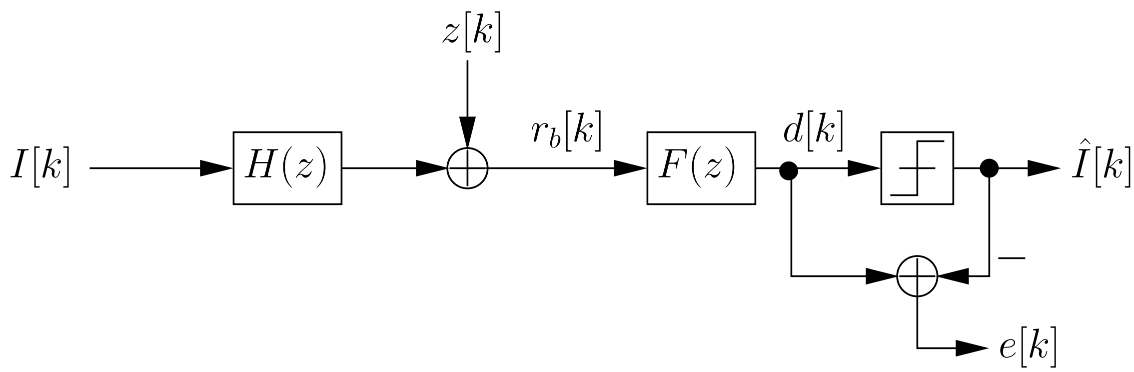
We observe that the residual interference is larger for shorter equalizer filter lengths and increases as the root of $H(z)$ approaches the unit circle.

6.3.3 Optimum Minimum Mean–Squared Error (MMSE) Equalization

- **Objective**

  Minimize the variance of the error signal

$$e[k] = d[k] - \hat{I}[k].$$



- **Advantage over ZF Equalization**

  The MMSE criterion ensures an optimum trade–off between residual ISI in $d[k]$ and noise enhancement. Therefore, MMSE equalizers achieve a significantly lower BEP compared to ZF equalizers at low–to–moderate SNRs.

- **Calculation of Optimum Filter $F(z)$**

  – The error signal $e[k] = d[k] - \hat{I}[k]$ depends on the *estimated* symbols $\hat{I}[k]$. Since it is very difficult to take into account the effect of possibly erroneous decisions, for filter optimization it is usually assumed that $\hat{I}[k] = I[k]$ is valid. The corresponding error signal is

$$e[k] = d[k] - I[k].$$

  – **Cost Function**
  The cost function for filter optimization is given by

$$\begin{aligned} J &= \mathcal{E}\{|e[k]|^2\} \\ &= \mathcal{E}\left\{\left(\sum_{m=-\infty}^{\infty} f[m]r_b[k-m] - I[k]\right)\right. \\ &\qquad \left.\left(\sum_{m=-\infty}^{\infty} f^*[m]r_b^*[k-m] - I^*[k]\right)\right\}, \end{aligned}$$

  which is the error variance.

  – **Optimum Filter**
  We obtain the optimum filter coefficients from

$$\begin{aligned} \frac{\partial J}{\partial f^*[\kappa]} &= \mathcal{E}\left\{\left(\sum_{m=-\infty}^{\infty} f[m]r_b[k-m] - I[k]\right) r_b^*[k-\kappa]\right\} \\ &= \mathcal{E}\left\{e[k]r_b^*[k-\kappa]\right\} = 0, \qquad \kappa \in \{\ldots, -1, 0, 1, \ldots\}, \end{aligned}$$

  where we have used the following rules for *complex differen-*

tiation

$$\begin{aligned}
\frac{\partial f^*[\kappa]}{\partial f^*[\kappa]} &= 1 \\
\frac{\partial f[\kappa]}{\partial f^*[\kappa]} &= 0 \\
\frac{\partial |f[\kappa]|^2}{\partial f^*[\kappa]} &= f[\kappa].
\end{aligned}$$

We observe that the error signal and the input of the MMSE filter must be *orthogonal*. This is referred to as the *orthogonality principle* of MMSE optimization.

The above condition can be modified to

$$\mathcal{E}\left\{e[k]r_b^*[k-\kappa]\right\} = \mathcal{E}\left\{d[k]r_b^*[k-\kappa]\right\} - \mathcal{E}\left\{I[k]r_b^*[k-\kappa]\right\}$$

The individual terms on the right hand side of the above equation can be further simplified to

$$\mathcal{E}\left\{d[k]r_b^*[k-\kappa]\right\} = \sum_{m=-\infty}^{\infty} f[m] \underbrace{\mathcal{E}\left\{r_b[k-m]r_b^*[k-\kappa]\right\}}_{\phi_{rr}[\kappa-m]},$$

and

$$\begin{aligned}
\mathcal{E}\left\{I[k]r_b^*[k-\kappa]\right\} &= \sum_{m=-\infty}^{\infty} h^*[m] \underbrace{\mathcal{E}\left\{I[k]I^*[k-\kappa-m]\right\}}_{\phi_{II}[\kappa+m]} \\
&= \sum_{\mu=-\infty}^{\infty} h^*[-\mu]\phi_{II}[\kappa-\mu],
\end{aligned}$$

respectively. Therefore, we obtain

$$f[k] * \phi_{rr}[k] = h^*[-k] * \phi_{II}[k],$$

and the $\mathcal{Z}$–transform of this equation is

$$F(z)\Phi_{rr}(z) = H^*(1/z^*)\Phi_{II}(z)$$

with

$$\begin{aligned}\Phi_{rr}(z) &= \sum_{k=-\infty}^{\infty} \phi_{rr}[k]z^{-k} \\ \Phi_{II}(z) &= \sum_{k=-\infty}^{\infty} \phi_{II}[k]z^{-k}.\end{aligned}$$

The optimum filter transfer function is given by

$$F(z) = \frac{H^*(1/z^*)\Phi_{II}(z)}{\Phi_{rr}(z)}$$

Usually, we assume that the noise $z[k]$ and the data sequence $I[k]$ are white processes and mutually uncorrelated. We assume furthermore that the variance of $I[k]$ is normalized to 1. In that case, we get

$$\begin{aligned}\phi_{rr}[k] &= h[k] * h^*[-k] * \phi_{II}[k] + \phi_{ZZ}[k] \\ &= h[k] * h^*[-k] + N_0\,\delta[k],\end{aligned}$$

and

$$\begin{aligned}\Phi_{rr}(z) &= H(z)H^*(1/z^*) + N_0 \\ \Phi_{II}(z) &= 1.\end{aligned}$$

The optimum MMSE filter is given by

$$\boxed{F(z) = \frac{H^*(1/z^*)}{H(z)H^*(1/z^*) + N_0}}$$

We may consider two limiting cases.

1. $N_0 \to 0$
   In this case, we obtain
   $$F(z) = \frac{1}{H(z)},$$
   i.e., in the high SNR region the MMSE solution approaches the ZF equalizer.
2. $N_0 \to \infty$
   We get
   $$F(z) = \frac{1}{N_0} H^*(1/z^*),$$
   i.e., the MMSE filter approaches a discrete–time matched filter.

- **Autocorrelation of Error Sequence**

The ACF of the error sequence $e[k]$ is given by

$$\begin{aligned}\phi_{ee}[\lambda] &= \mathcal{E}\{e[k]e^*[k-\lambda]\}\\ &= \mathcal{E}\left\{e[k](d[k-\lambda] - I[k-\lambda])^*\right\}\\ &= \phi_{ed}[\lambda] - \phi_{eI}[\lambda]\end{aligned}$$

$\phi_{ed}[\lambda]$ can be simplified to

$$\begin{aligned}\phi_{ed}[\lambda] &= \sum_{m=-\infty}^{\infty} f^*[m] \underbrace{\mathcal{E}\{e[k]r_b^*[k-\lambda-m]\}}_{=0}\\ &= 0.\end{aligned}$$

This means that the error signal $e[k]$ is also orthogonal to the equalizer output signal $d[k]$. For the ACF of the error we obtain

$$\begin{aligned}\phi_{ee}[\lambda] &= -\phi_{eI}[\lambda]\\ &= \phi_{II}[\lambda] - \phi_{dI}[\lambda].\end{aligned}$$

- **Error Variance**

The error variance σ_e^2 is given by

$$\begin{aligned}\sigma_e^2 &= \phi_{II}[0] - \phi_{dI}[0] \\ &= 1 - \phi_{dI}[0].\end{aligned}$$

σ_e^2 can be calculated most easily from the the power spectral density

$$\begin{aligned}\Phi_{ee}(z) &= \Phi_{II}(z) - \Phi_{dI}(z) \\ &= 1 - F(z)H(z) \\ &= 1 - \frac{H(z)H^*(1/z^*)}{H(z)H^*(1/z^*) + N_0} \\ &= \frac{N_0}{H(z)H^*(1/z^*) + N_0}.\end{aligned}$$

More specifically, σ_e^2 is given by

$$\sigma_e^2 = T \int\limits_{-1/(2T)}^{1/(2T)} \Phi_{ee}(e^{j2\pi fT})\,\mathrm{d}f$$

or

$$\boxed{\sigma_e^2 = T \int\limits_{-1/(2T)}^{1/(2T)} \frac{N_0}{|H(e^{j2\pi fT})|^2 + N_0}\,\mathrm{d}f}$$

Overall Transfer Function

The overall transfer function is given by

$$\begin{aligned} H_{\text{ov}}(z) &= H(z)F(z) \\ &= \frac{H(z)H^*(1/z^*)}{H(z)H^*(1/z^*) + N_0} \\ &= \frac{1}{1 + \frac{N_0}{H(z)H^*(1/z^*)}} \\ &= 1 - \frac{N_0}{H(z)H^*(1/z^*) + N_0} \end{aligned}$$

Obviously, $H_{\text{ov}}(z)$ is not a constant but depends on z, i.e., there is residual intersymbol interference. The coefficient $h_{\text{ov}}[0]$ is obtained from

$$\begin{aligned} h_{\text{ov}}[0] &= T \int_{-1/(2T)}^{1/(2T)} H_{\text{ov}}(e^{j2\pi fT})\,\mathrm{d}f \\ &= T \int_{-1/(2T)}^{1/(2T)} (1 - \Phi_{ee}(e^{j2\pi fT}))\,\mathrm{d}f \\ &= 1 - \sigma_e^2 < 1. \end{aligned}$$

Since $h_{\text{ov}}[0] < 1$ is valid, MMSE equalization is said to be *biased*.

SNR

The decision variable $d[k]$ may be rewritten as

$$\begin{aligned} d[k] &= I[k] + e[k] \\ &= h_{\text{ov}}[0]I[k] + \underbrace{e[k] + (1 - h_{\text{ov}}[0])I[k]}_{=e'[k]}, \end{aligned}$$

where $e'[k]$ does not contain $I[k]$. Using $\phi_{ee}[\lambda] = -\phi_{eI}[\lambda]$ the variance of $e'[k]$ is given by

$$\begin{aligned}
\sigma^2_{e'} &= \mathcal{E}\{|e[k]|^2\} \\
&= (1 - h_{\mathrm{ov}}[0])^2 + 2(1 - h_{\mathrm{ov}}[0])\phi_{eI}[0] + \sigma^2_e \\
&= \sigma^4_e - 2\sigma^4_e + \sigma^2_e \\
&= \sigma^2_e - \sigma^4_e.
\end{aligned}$$

Therefore, the SNR for MMSE equalization with IIR filters is given by

$$\begin{aligned}
\mathrm{SNR}_{\mathrm{IIR-MMSE}} &= \frac{h^2_{\mathrm{ov}}[0]}{\sigma^2_{e'}} \\
&= \frac{(1 - \sigma^2_e)^2}{\sigma^2_e(1 - \sigma^2_e)}
\end{aligned}$$

which yields

$$\boxed{\mathrm{SNR}_{\mathrm{IIR-MMSE}} = \frac{1 - \sigma^2_e}{\sigma^2_e}}$$

Example:

We consider again the channel with one root and transfer function

$$H(z) = \frac{1}{\sqrt{1+|c|^2}}\,(1 - cz^{-1}),$$

where c is a complex constant. After some straightforward manipulations the error variance is given by

$$\begin{aligned}\sigma_e^2 &= \mathcal{E}\{|e[k]|^2\} \\ &= T \int\limits_{-1/(2T)}^{1/(2T)} \frac{N_0}{|H(e^{j2\pi fT})|^2 + N_0}\,\mathrm{d}f \\ &= \frac{N_0}{1+N_0}\,\frac{1}{\sqrt{1-\beta^2}},\end{aligned}$$

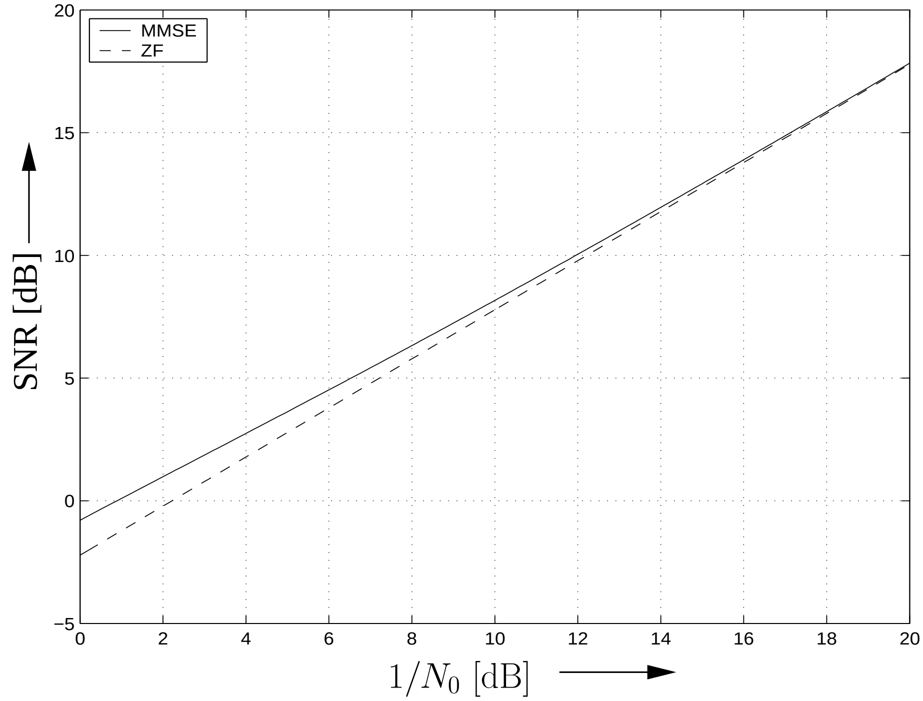
where β is defined as

$$\beta = \frac{2|c|}{(1+N_0)(1+|c|^2)}.$$

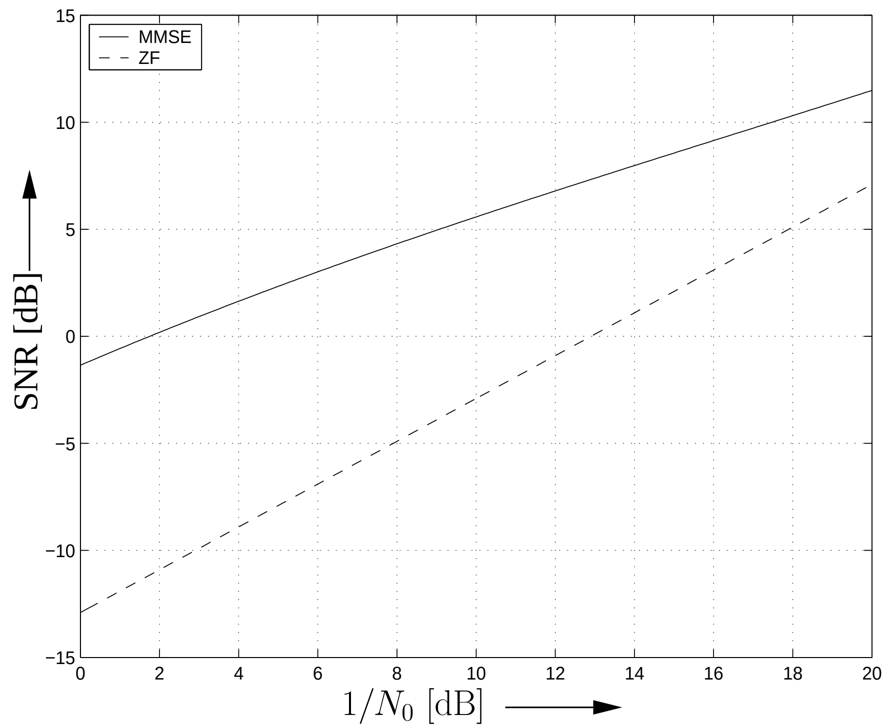
It is easy to check that for $N_0 \to 0$, i.e., for high SNRs σ_e^2 approaches the error variance for linear ZF equalization.

We illustrate the SNR for two different cases.

1. $|c| = 0.5$


MMSE
ZF
SNR [dB]
1/N0 [dB]
20
15
10
5
0
−5
0
2
4
6
8
10
12
14
16
18
20

2. $|c| = 0.95$


MMSE
ZF
SNR [dB]
1/N0 [dB]
15
10
5
0
−5
−10
−15
0
2
4
6
8
10
12
14
16
18
20

As expected, for high input SNRs (small noise variances), the (output) SNR for ZF equalization approaches that for MMSE equalization. For larger noise variances, however, MMSE equalization yields a significantly higher SNR, especially if $H(z)$ has zeros close to the unit circle.

6.3.4 MMSE Equalization with FIR Filters

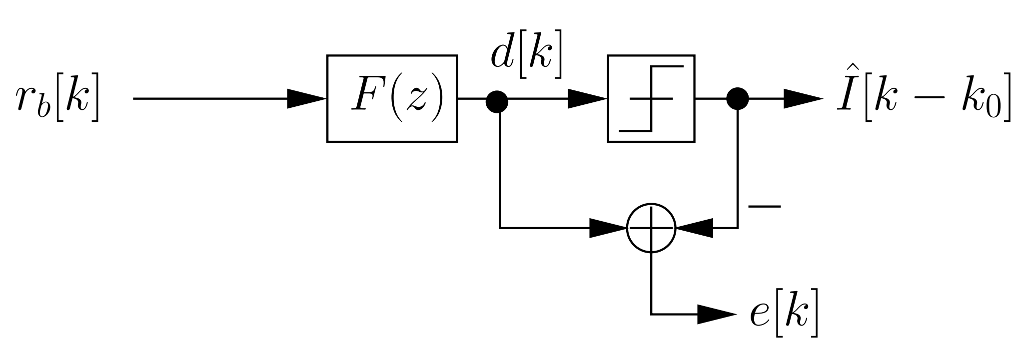
- In practice, FIR filters are employed. The equalizer output signal in that case is given by

$$\begin{aligned} d[k] &= \sum_{m=0}^{q_F} f[m] r_b[k-m] \\ &= \boldsymbol{f}^H \boldsymbol{r}_b, \end{aligned}$$

where $q_F = L_F - 1$ and the definitions

$$\begin{aligned} \boldsymbol{f} &= [f[0] \ \ldots \ f[q_F]]^H \\ \boldsymbol{r}_b &= [r_b[k] \ \ldots \ r_b[k-q_F]]^T \end{aligned}$$

are used. Note that vector $\boldsymbol{f}$ contains the *complex conjugate* filter coefficients. This is customary in the literature and simplifies the derivation of the optimum filter coefficients.



Error Signal

The error signal $e[k]$ is given by

$$e[k] = d[k] - I[k - k_0],$$

where we allow for a decision delay $k_0 \geq 0$ to account for non–causal components, and we assume again $\hat{I}[k - k_0] = I[k - k_0]$ for the sake of mathematical tractability.

Cost Function

The cost function for filter optimization is given by

$$\begin{aligned} J(\boldsymbol{f}) &= \mathcal{E}\{|e[k]|^2\} \\ &= \mathcal{E}\left\{\left(\boldsymbol{f}^H \boldsymbol{r}_b - I[k-k_0]\right)\left(\boldsymbol{f}^H \boldsymbol{r}_b - I[k-k_0]\right)^H\right\} \\ &= \boldsymbol{f}^H \underbrace{\mathcal{E}\{\boldsymbol{r}_b \boldsymbol{r}_b^H\}}_{\boldsymbol{\Phi}_{rr}} \boldsymbol{f} - \boldsymbol{f}^H \underbrace{\mathcal{E}\{\boldsymbol{r}_b I^*[k-k_0]\}}_{\boldsymbol{\varphi}_{rI}} \\ &\quad - \mathcal{E}\{I[k-k_0]\boldsymbol{r}_b^H\}\boldsymbol{f} + \mathcal{E}\{|I[k-k_0]|^2\} \\ &= \boldsymbol{f}^H \boldsymbol{\Phi}_{rr} \boldsymbol{f} - \boldsymbol{f}^H \boldsymbol{\varphi}_{rI} - \boldsymbol{\varphi}_{rI}^H \boldsymbol{f} + 1, \end{aligned}$$

where $\boldsymbol{\Phi}_{rr}$ denotes the autocorrelation matrix of vector $\boldsymbol{r}_b$, and $\boldsymbol{\varphi}_{rI}$ is the crosscorrelation vector between $\boldsymbol{r}_b$ and $I[k - k_0]$. $\boldsymbol{\Phi}_{rr}$ is given by

$$\boldsymbol{\Phi}_{rr} = \begin{bmatrix} \phi_{rr}[0] & \phi_{rr}[1] & \cdots & \phi_{rr}[q_F] \\ \phi_{rr}[-1] & \phi_{rr}[0] & \cdots & \phi_{rr}[q_F - 1] \\ \vdots & \vdots & \ddots & \vdots \\ \phi_{rr}[-q_F] & \phi_{rr}[-q_F + 1] & \cdots & \phi_{rr}[0] \end{bmatrix},$$

where $\phi_{rr}[\lambda] = \mathcal{E}\{r_b^*[k] r_b[k + \lambda]\}$. The crosscorrelation vector can be calculated as

$$\boldsymbol{\varphi}_{rI} = [\phi_{rI}[k_0] \; \phi_{rI}[k_0 - 1] \; \ldots \; \phi_{rI}[k_0 - q_F]]^T,$$

where $\phi_{rI}[\lambda] = \mathcal{E}\{r_b[k+\lambda]I^*[k]\}$. Note that for independent, identically distributed input data and AWGN, we get

$$\begin{aligned}\phi_{rr}[\lambda] &= h[\lambda] * h^*[-\lambda] + N_0\,\delta[\lambda] \\ \phi_{rI}[\lambda] &= h[k_0+\lambda].\end{aligned}$$

This completely specifies $\boldsymbol{\Phi}_{rr}$ and $\boldsymbol{\varphi}_{rI}$.

- **Filter Optimization**

The optimum filter coefficient vector can be obtained be setting the gradient of $J(\boldsymbol{f})$ equal to zero

$$\frac{\partial J(\boldsymbol{f})}{\partial \boldsymbol{f}^*} = \boldsymbol{0}.$$

For calculation of this gradient, we use the following rules for differentiation of scalar functions with respect to (complex) vectors:

$$\begin{aligned}\frac{\partial}{\partial \boldsymbol{f}^*}\boldsymbol{f}^H\boldsymbol{X}\boldsymbol{f} &= \boldsymbol{X}\boldsymbol{f} \\ \frac{\partial}{\partial \boldsymbol{f}^*}\boldsymbol{f}^H\boldsymbol{x} &= \boldsymbol{x} \\ \frac{\partial}{\partial \boldsymbol{f}^*}\boldsymbol{x}^H\boldsymbol{f} &= \boldsymbol{0},\end{aligned}$$

where $\boldsymbol{X}$ and $\boldsymbol{x}$ denote a matrix and a vector, respectively.

With these rules we obtain

$$\frac{\partial J(\boldsymbol{f})}{\partial \boldsymbol{f}^*} = \boldsymbol{\Phi}_{rr}\boldsymbol{f} - \boldsymbol{\varphi}_{rI} = \boldsymbol{0}$$

or

$$\boldsymbol{\Phi}_{rr}\boldsymbol{f} = \boldsymbol{\varphi}_{rI}.$$

This equation is often referred to as the *Wiener–Hopf equation.* The MMSE or Wiener solution for the optimum filter coefficients $\boldsymbol{f}_{\text{opt}}$ is given by

$$\boxed{\boldsymbol{f}_{\text{opt}} = \boldsymbol{\Phi}_{rr}^{-1}\boldsymbol{\varphi}_{rI}}$$

- **Error Variance**

  The minimum error variance is given by

$$\begin{aligned}\sigma_e^2 &= J(\boldsymbol{f}_{\text{opt}}) \\ &= 1 - \boldsymbol{\varphi}_{rI}^H\boldsymbol{\Phi}_{rr}^{-1}\boldsymbol{\varphi}_{rI} \\ &= 1 - \boldsymbol{\varphi}_{rI}^H\boldsymbol{f}_{\text{opt}}\end{aligned}$$

- **Overall Channel Coefficient** $h_{\text{ov}}[k_0]$

  The coefficient $h_{\text{ov}}[k_0]$ is given by

$$\begin{aligned}h_{\text{ov}}[k_0] &= \boldsymbol{f}_{\text{opt}}^H\boldsymbol{\varphi}_{rI} \\ &= 1 - \sigma_e^2 < 1,\end{aligned}$$

  i.e., also the optimum FIR MMSE filter is *biased.*

- **SNR**

  Similar to the IIR case, the SNR at the output of the optimum FIR filter can be calculated to

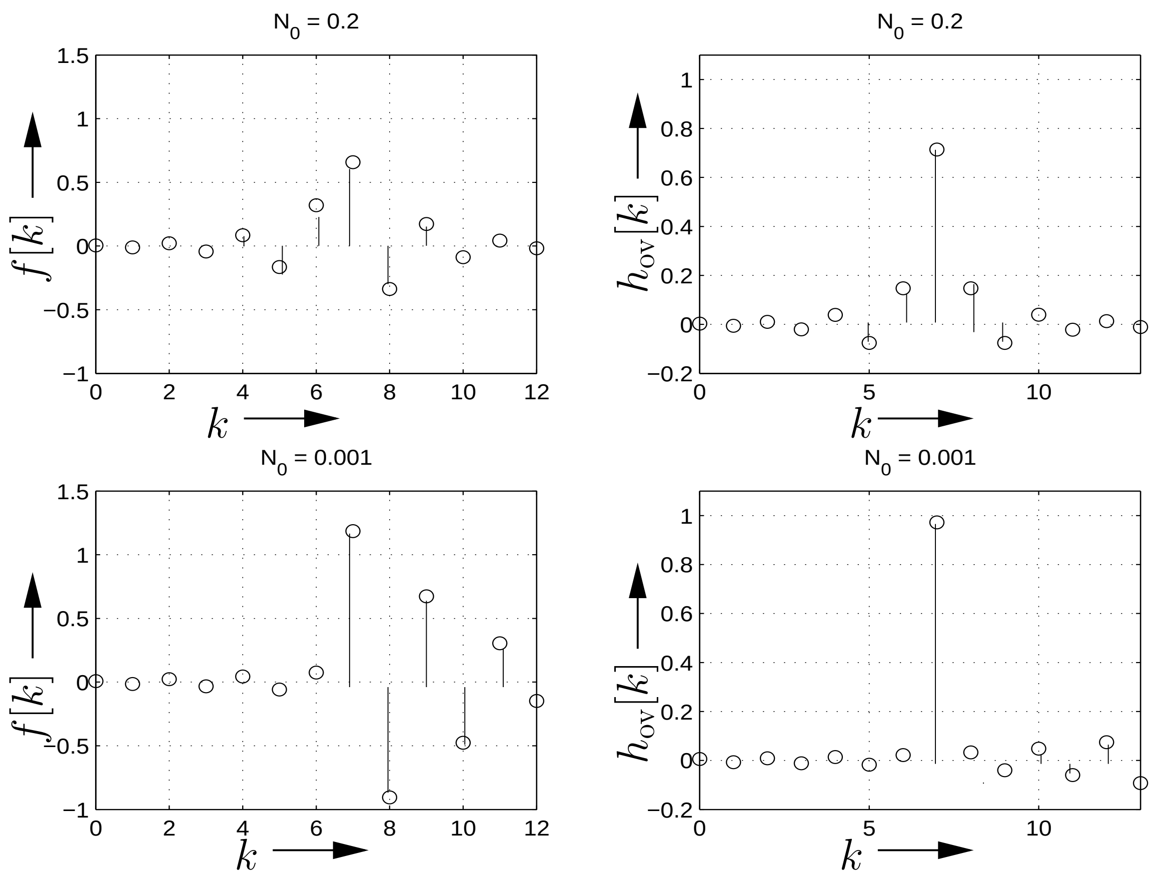
$$\boxed{\text{SNR}_{\text{FIR-MMSE}} = \frac{1-\sigma_e^2}{\sigma_e^2}}$$

Example:

Below, we show $f[k]$ and $h_{\mathrm{ov}}[k]$ for a channel with one root and transfer function

$$H(z) = \frac{1}{\sqrt{1+|c|^2}}(1 - cz^{-1}).$$

We consider the case $c = 0.8$ and different noise variances $\sigma_Z^2 = N_0$. Furthermore, we use $q_F = 12$ and $k_0 = 7$.

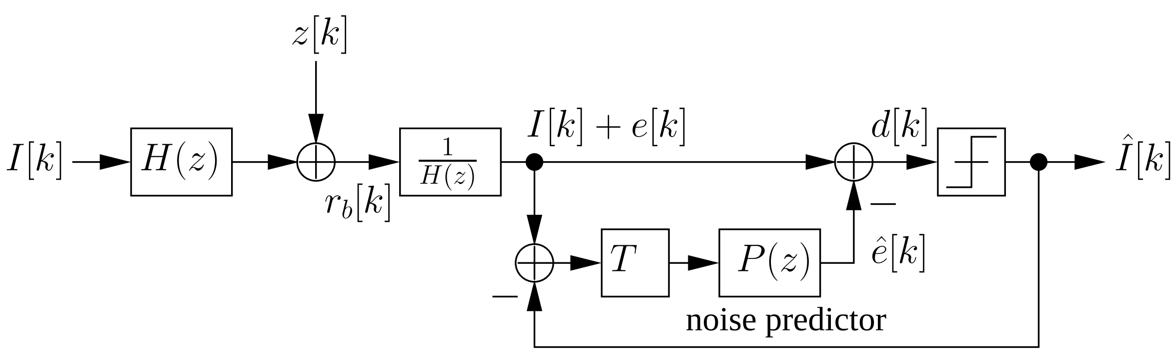


We observe that the residual ISI in $h_{\mathrm{ov}}[k]$ is smaller for the smaller noise variance, since the MMSE filter approaches the ZF filter for $N_0 \rightarrow 0$. Also the bias decreases with decreasing noise variance, i.e., $h_{\mathrm{ov}}[k_0]$ approaches 1.

6.4 Decision–Feedback Equalization (DFE)

- **Drawback of Linear Equalization**
  In linear equalization, the equalizer filter enhances the noise, especially in severely distorted channels with roots close to the unit circle. The noise variance at the equalizer output is *increased* and the noise is *colored*. In many cases, this leads to a poor performance.

- **Noise Prediction**
  The above described drawback of LE can be avoided by application of *linear noise prediction*.



The linear FIR noise predictor

$$P(z) = \sum_{m=0}^{L_P-1} p[m]z^{-m}$$

predicts the current noise sample $e[k]$ based on the previous L_P noise samples $e[k-1]$, $e[k-2]$, $\ldots$, $e[k-L_P]$. The estimate $\hat{e}[k]$ for $e[k]$ is given by

$$\hat{e}[k] = \sum_{m=0}^{L_P-1} p[m]e[k-1-m].$$

Assuming $\hat{I}[k-m] = I[k-m]$ for $m \geq 1$, the new decision variable is

$$d[k] = I[k] + e[k] - \hat{e}[k].$$

If the predictor coefficients are suitably chosen, we expect that the variance of the new error signal $e[k] - \hat{e}[k]$ is smaller than that of $e[k]$. Therefore, noise prediction improves performance.

- **Predictor Design**

  Usually an MMSE criterion is adopted for optimization of the predictor coefficients, i.e., the design objective is to minimize the error variance

$$\mathcal{E}\{|e[k] - \hat{e}[k]|^2\}.$$

Since this is a typical MMSE problem, the optimum predictor coefficients can be obtained from the Wiener–Hopf equation

$$\boldsymbol{\Phi}_{ee}\boldsymbol{p} = \boldsymbol{\varphi}_e$$

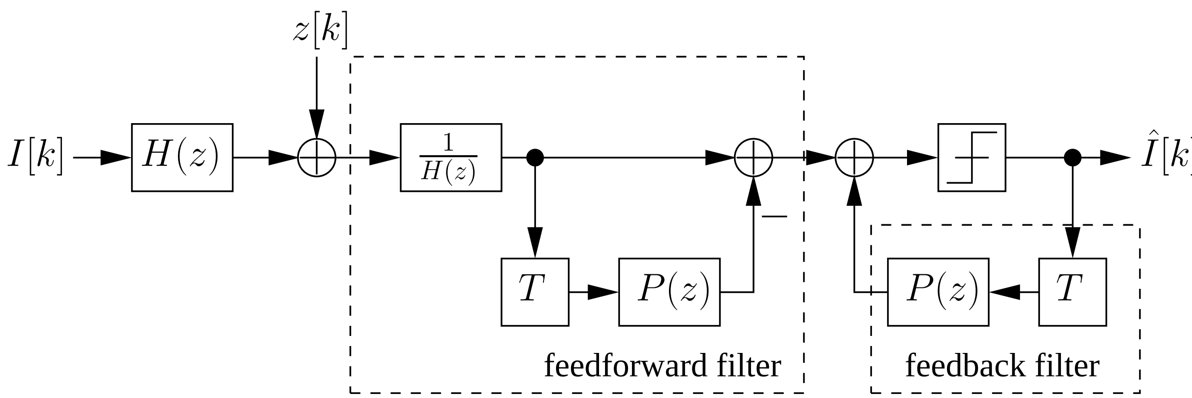
with

$$\begin{aligned}
\boldsymbol{\Phi}_{ee} &= \begin{bmatrix} \phi_{ee}[0] & \phi_{ee}[1] & \cdots & \phi_{ee}[L_P-1] \\ \phi_{ee}[-1] & \phi_{ee}[0] & \cdots & \phi_{ee}[L_P-2] \\ \vdots & \vdots & \ddots & \vdots \\ \phi_{ee}[-(L_P-1)] & \phi_{ee}[-(L_P-2)] & \cdots & \phi_{ee}[0] \end{bmatrix}, \\
\boldsymbol{\varphi}_e &= [\phi_{ee}[-1]\ \phi_{ee}[-2]\ \ldots\ \phi_{ee}[-L_P]]^T \\
\boldsymbol{p} &= [p[0]\ p[1]\ \ldots\ p[L_P-1]]^H,
\end{aligned}$$

where the ACF of $e[k]$ is defined as $\phi_{ee}[\lambda] = \mathcal{E}\{e^*[k]e[k+\lambda]\}$.

■ New Block Diagram

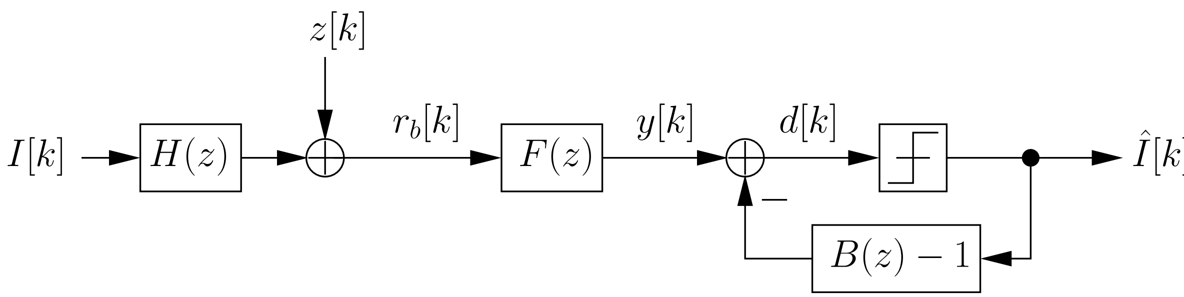
The above block diagram of linear equalization and noise prediction can be rearranged as follows.



The above structure consists of two filters. A *feedforward filter* whose input is the channel output signal $r_b[k]$ and a *feedback filter* that feeds back previous decisions $\hat{I}[k-m]$, $m \geq 1$. An equalization scheme with this structure is referred to as *decision–feedback equalization (DFE)*. We have shown that the DFE structure can be obtained in a natural way from linear equalization and noise prediction.

■ General DFE

If the predictor has infinite length, the above DFE scheme corresponds to optimum zero–forcing (ZF) DFE. However, the DFE concept can be generalized of course allowing for different filter optimization criteria. The structure of a general DFE scheme is shown below.



In general, the feedforward filter is given by

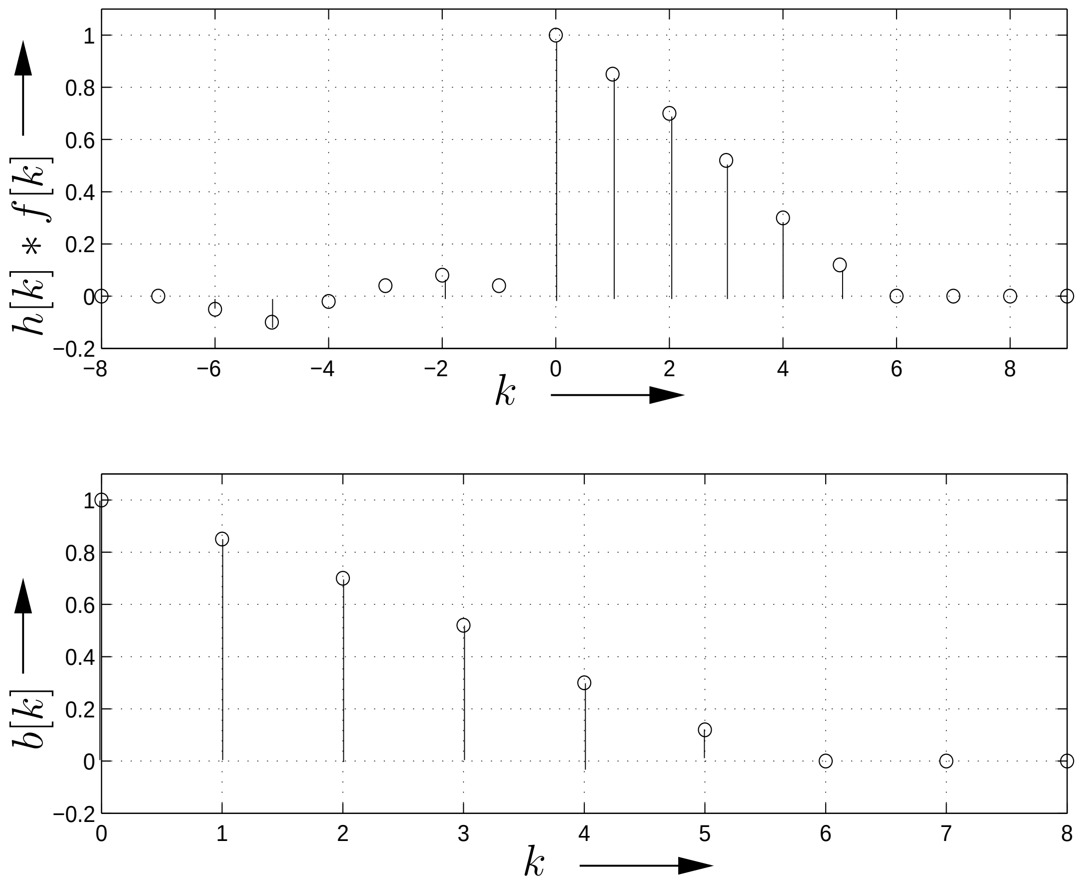
$$F(z) = \sum_{k=-\infty}^{\infty} f[k] z^{-k},$$

and the feedback filter

$$B(z) = 1 + \sum_{k=1}^{L_B-1} b[k] z^{-k}$$

is *causal* and *monic* ($b[k] = 1$). Note that $F(z)$ may also be an FIR filter. $F(z)$ and $B(z)$ can be optimized according to any suitable criterion, e.g. ZF or MMSE criterion.

Typical Example:



- **Properties of DFE**

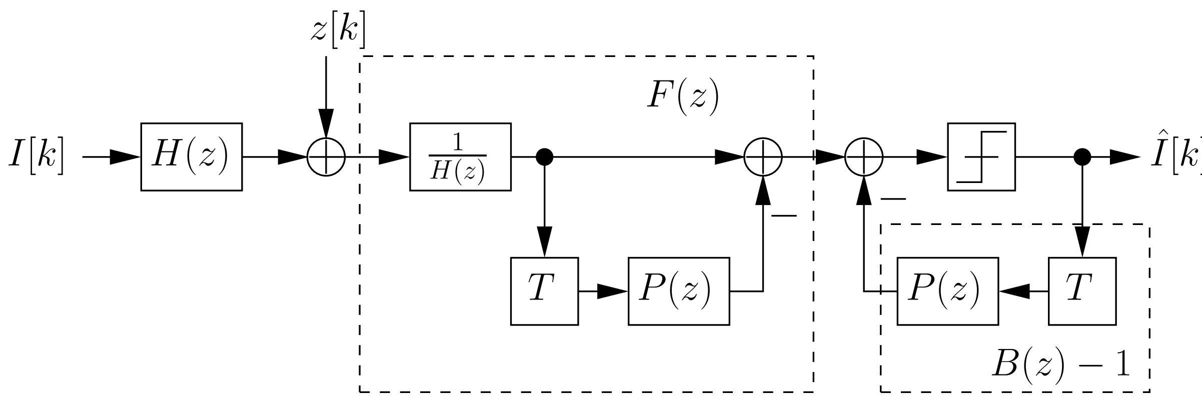
  – The feedforward filter has to suppress only the pre–cursor ISI. This imposes fewer constraints on the feedforward filter and therefore, the noise enhancement for DFE is significantly smaller than for linear equalization.

  – The post–cursors are canceled by the feedback filter. This causes no additional noise enhancement since the slicer eliminates the noise before feedback.

  – Feedback of wrong decisions causes error propagation. Fortunately, this error propagation is usually not catastrophic but causes some performance degradation compared to error free feedback.

6.4.1 Optimum ZF–DFE

- Optimum ZF–DFE may be viewed as a combination of optimum linear ZF equalization and optimum noise prediction.



- **Equalizer Filters (I)**
  The feedforward filter (FFF) is the cascade of the linear equalizer $1/H(z)$ and the *prediction error filter* $P_e(z) = 1 - z^{-1}P(z)$

$$F(z) = \frac{P_e(z)}{H(z)}.$$

The feedback filter (FBF) is given by

$$B(z) = 1 - z^{-1}P(z).$$

- **Power Spectral Density of Noise**
  The power spectral density of the noise component $e[k]$ is given by

$$\Phi_{ee}(z) = \frac{N_0}{H(z)H^*(1/z^*)}$$

Optimum Noise Prediction

The optimum noise prediction error filter is a *noise whitening filter*, i.e., the power spectrum $\Phi_{\nu\nu}(z)$ of $\nu[k] = e[k] - \hat{e}[k]$ is a constant

$$\Phi_{\nu\nu}(z) = P_e(z)P_e^*(1/z^*)\Phi_{ee}(z) = \sigma_\nu^2,$$

where σ_ν^2 is the variance of ν. A more detailed analysis shows that $P_e(z)$ is given by

$$P_e(z) = \frac{1}{Q(z)}.$$

$Q(z)$ is monic and stable, and is obtained by *spectral factorization* of $\Phi_{ee}(z)$ as

$$\Phi_{ee}(z) = \sigma_\nu^2\, Q(z)Q^*(1/z^*). \tag{5}$$

Furthermore, we have

$$\begin{aligned} \Phi_{ee}(z) &= \frac{N_0}{H(z)H^*(1/z^*)} \\ &= \frac{N_0}{H_{\min}(z)H^*_{\min}(1/z^*)}, \end{aligned} \tag{6}$$

where

$$H_{\min}(z) = \sum_{m=0}^{L-1} h_{\min}[m]z^{-m}$$

is the minimum phase equivalent of $H(z)$, i.e., we get $H_{\min}(z)$ from $H(z)$ by mirroring all zeros of $H(z)$ that are outside the unit circle into the unit circle. A comparison of Eqs. (5) and (6) shows that $Q(z)$ is given by

$$Q(z) = \frac{h_{\min}[0]}{H_{\min}(z)},$$

where the multiplication by $h_{\min}[0]$ ensures that $Q(z)$ is monic. Since $H_{\min}(z)$ is minimum phase, all its zeros are inside or on the unit circle, therefore $Q(z)$ is stable. The prediction error variance is given by

$$\sigma_\nu^2 = \frac{N_0}{|h_{\min}[0]|^2}.$$

The optimum noise prediction–error filter is obtained as

$$P_e(z) = \frac{H_{\min}(z)}{h_{\min}[0]}.$$

- **Equalizer Filters (II)**
  With the above result for $P_e(z)$, the optimum ZF–DFE FFF is given by

$$\begin{aligned} F(z) &= \frac{P_e(z)}{H(z)} \\ F(z) &= \frac{1}{h_{\min}[0]} \frac{H_{\min}(z)}{H(z)}, \end{aligned}$$

whereas the FBF is obtained as

$$\begin{aligned} B(z) &= 1 - z^{-1}P(z) = 1 - (1 - P_e(z)) \\ &= P_e(z) \\ &= \frac{H_{\min}(z)}{h_{\min}[0]}. \end{aligned}$$

- **Overall Channel**

  The overall forward channel is given by

$$\begin{aligned} H_{\text{ov}}(z) &= H(z)F(z) \\ &= \frac{H_{\min}(z)}{h_{\min}[0]} \\ &= \sum_{m=0}^{L-1} \frac{h_{\min}[m]}{h_{\min}[0]} z^{-m} \end{aligned}$$

  This means the FFF filter $F(z)$ transforms the channel $H(z)$ into its (scaled) minimum phase equivalent. The FBF is given by

$$\begin{aligned} B(z) - 1 &= \frac{H_{\min}(z)}{h_{\min}[0]} - 1 \\ &= \sum_{m=1}^{L-1} \frac{h_{\min}[m]}{h_{\min}[0]} z^{-m} \end{aligned}$$

  Therefore, assuming error–free feedback the equivalent overall channel including forward and backward part is an ISI–free channel with gain 1.

- **Noise**

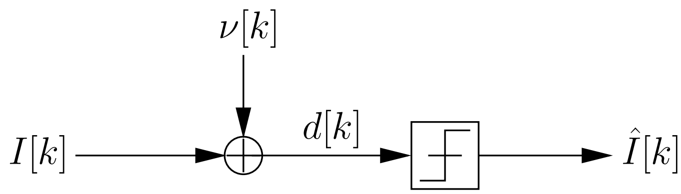
  The FFF $F(z)$ is an allpass filter since

$$\begin{aligned} F(z)F^*(1/z^*) &= \frac{1}{|h_{\min}[0]|^2} \frac{H_{\min}(z)H_{\min}^*(1/z^*)}{H(z)H^*(1/z^*)} \\ &= \frac{1}{|h_{\min}[0]|^2}. \end{aligned}$$

  Therefore, the noise component $\nu[k]$ is AWGN with variance

$$\sigma_\nu^2 = \frac{N_0}{|h_{\min}[0]|^2}.$$

The equivalent overall channel model assuming error–free feedback is shown below.



- **SNR**

Obviously, the SNR of optimum ZF–DFE is given by

$$\begin{aligned} \mathrm{SNR}_{\mathrm{ZF-DFE}} &= \frac{1}{\sigma_\nu^2} \\ &= \frac{|h_{\min}[0]|^2}{N_0}. \end{aligned}$$

Furthermore, it can be shown that $h_{\min}[0]$ can be calculated in closed form as a function of $H(z)$. This leads to

$$\boxed{\mathrm{SNR}_{\mathrm{ZF-DFE}} = \exp\left(T \int\limits_{-1/(2T)}^{1/(2T)} \ln\left(\frac{|H(e^{j2\pi fT})|^2}{N_0} \right) \,\mathrm{d}f \right).}$$

Example:

– *Channel*

We assume again a channel with one root and transfer function

$$H(z) = \frac{1}{\sqrt{1+|c|^2}}\,(1 - cz^{-1}).$$

– *Normalized Minimum–Phase Equivalent*

If $|c| \leq 1$, $H(z)$ is already minimum phase and we get

$$\begin{aligned} P_e(z) &= \frac{H_{\min}(z)}{h_{\min}[0]} \\ &= 1 - cz^{-1}, \qquad\qquad |c| \leq 1. \end{aligned}$$

If $|c| > 1$, the root of $H(z)$ has to be mirrored into the unit circle. Therefore, $H_{\min}(z)$ will have a zero at $z = 1/c^*$, and we get

$$\begin{aligned} P_e(z) &= \frac{H_{\min}(z)}{h_{\min}[0]} \\ &= 1 - \frac{1}{c^*}\,z^{-1}, \qquad\qquad |c| > 1. \end{aligned}$$

– *Filters*

The FFF is given by

$$F(z) = \begin{cases} \frac{1}{\sqrt{1+|c|^2}}, & |c| \leq 1 \\ \frac{1}{\sqrt{1+|c|^2}} \frac{z-1/c^*}{z-c}, & |c| > 1 \end{cases}$$

The corresponding FBF is

$$B(z) - 1 = \begin{cases} -cz^{-1}, & |c| \leq 1 \\ -\frac{1}{c^*}\,z^{-1}, & |c| > 1 \end{cases}$$

– *SNR*

The SNR can be calculated to

$$\mathrm{SNR}_{\mathrm{ZF-DFE}} = \exp\left(T \int\limits_{-1/(2T)}^{1/(2T)} \ln\left(\frac{|H(e^{j2\pi fT})|^2}{N_0}\right) \mathrm{d}f\right).$$

$$= \exp\left(T \int\limits_{-1/(2T)}^{1/(2T)} \ln\left(\frac{|1 - ce^{-2\pi fT}|^2}{(1+|c|^2)N_0}\right) \mathrm{d}f\right).$$

After some straightforward manipulations, we obtain

$$\mathrm{SNR}_{\mathrm{ZF-DFE}} = \frac{1}{2N_0}\left(1 + \frac{|1-|c|^2|}{1+|c|^2}\right)$$

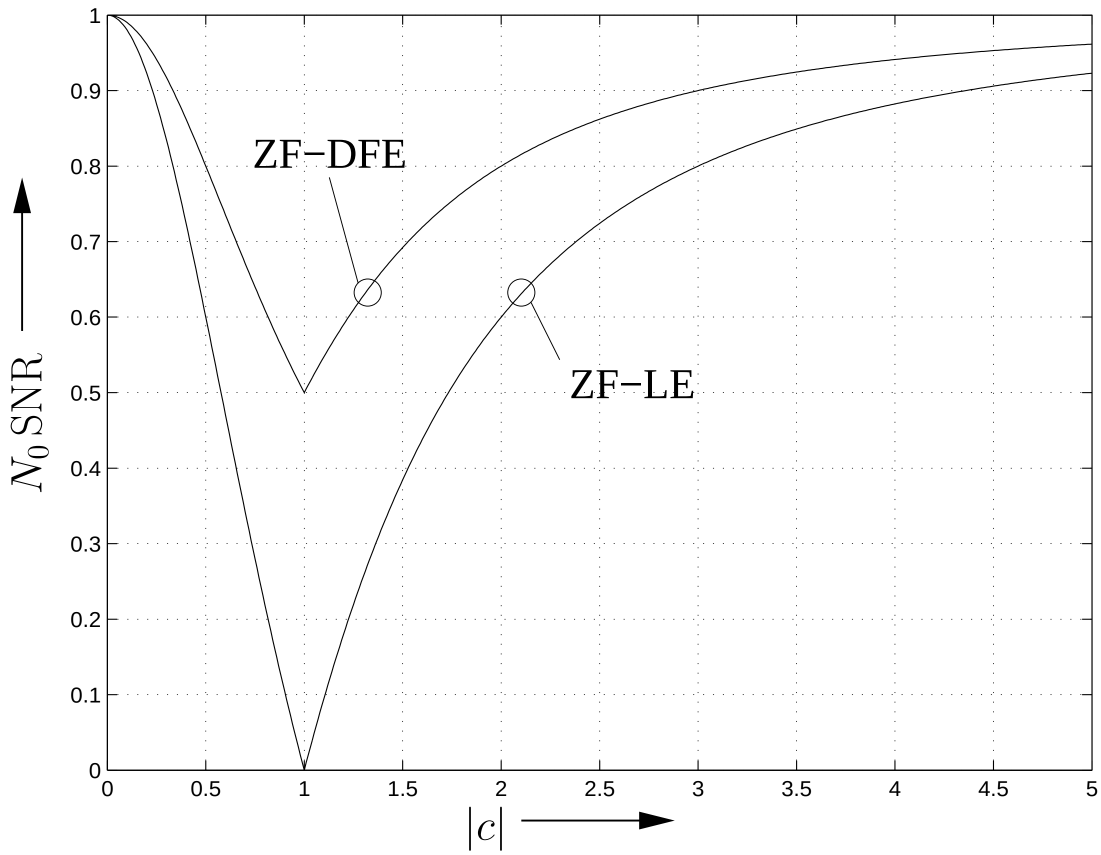
For a given N_0 the SNR is minimized for $|c| = 1$. In that case, we get $\mathrm{SNR}_{\mathrm{ZF-DFE}} = 1/(2N_0)$, i.e., there is a 3 dB loss compared to the pure AWGN channel. For $|c| = 0$ and $|c| \to \infty$, we get $\mathrm{SNR}_{\mathrm{ZF-DFE}} = 1/N_0$, i.e., there is no loss compared to the AWGN channel.

– *Comparison with ZF–LE*

For linear ZF equalization we had

$$\mathrm{SNR}_{\mathrm{ZF-LE}} = \frac{1}{N_0}\,\frac{|1-|c|^2|}{1+|c|^2}.$$

This means in the worst case $|c| = 1$, we get $\mathrm{SNR}_{\mathrm{ZF-LE}} = 0$ and reliable transmission is not possible. For $|c| = 0$ and $|c| \to \infty$ we obtain $\mathrm{SNR}_{\mathrm{ZF-LE}} = 1/N_0$ and no loss in performance is suffered compared to the pure AWGN channel.



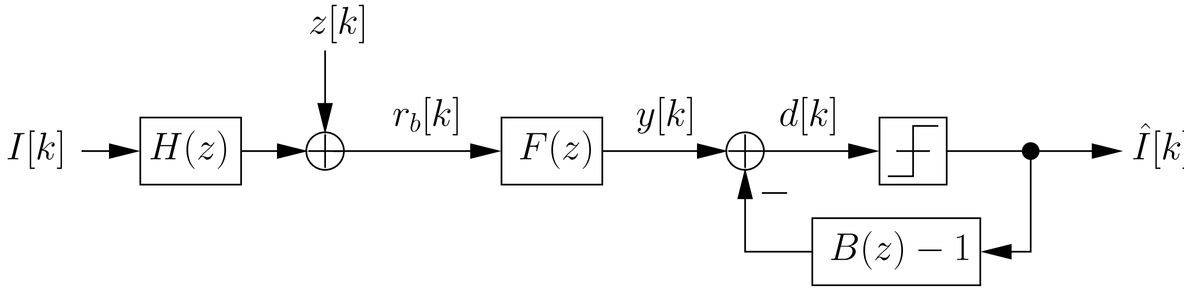
6.4.2 Optimum MMSE–DFE

- We assume a FFF with doubly–infinite response

$$F(z) = \sum_{k=-\infty}^{\infty} f[k]z^{-k}$$

and a causal FBF with

$$B(z) = 1 + \sum_{k=1}^{\infty} b[k]z^{-k}$$



- *Optimization Criterion*
  In optimum MMSE–DFE, we optimize FFF and FBF for minimization of the variance of the error signal

$$e[k] = d[k] - I[k].$$

  This error variance can be expressed as

$$\begin{aligned} J &= \mathcal{E}\left\{|e[k]|^2\right\} \\ &= \mathcal{E}\left\{\left(\sum_{\kappa=-\infty}^{\infty} f[\kappa]r_b[k-\kappa] - \sum_{\kappa=1}^{\infty} b[\kappa]I[k-\kappa] - I[k]\right)\right. \\ &\qquad \left.\left(\sum_{\kappa=-\infty}^{\infty} f^*[\kappa]r_b^*[k-\kappa] - \sum_{\kappa=1}^{\infty} b^*[\kappa]I^*[k-\kappa] - I^*[k]\right)\right\}. \end{aligned}$$

- **FFF Optimization**
  Differentiating J with respect to $f^*[\nu]$, $-\infty < \nu < \infty$, yields

$$\begin{aligned} \frac{\partial J}{\partial f^*[\nu]} &= \sum_{\kappa=-\infty}^{\infty} f[\kappa] \underbrace{\mathcal{E}\{r_b[k-\kappa]r_b^*[k-\nu]\}}_{\phi_{rr}[\nu-\kappa]} \\ &\quad - \sum_{\kappa=1}^{\infty} b[\kappa] \underbrace{\mathcal{E}\{I[k-\kappa]r_b^*[k-\nu]\}}_{\phi_{Ir}[\nu-\kappa]} - \underbrace{\mathcal{E}\{I[k]r_b^*[k-\nu]\}}_{\phi_{Ir}[\nu]} \end{aligned}$$

Letting $\partial J/(\partial f^*[\nu]) = 0$ and taking the $\mathcal{Z}$-transform of the above equation leads to

$$F(z)\Phi_{rr}(z) = B(z)\Phi_{Ir}(z),$$

where $\Phi_{rr}(z)$ and $\Phi_{Ir}(z)$ denote the $\mathcal{Z}$-transforms of $\phi_{rr}[\lambda]$ and $\phi_{Ir}[\lambda]$, respectively.

Assuming i.i.d. sequences $I[\cdot]$ of unit variance, we get

$$\begin{aligned} \Phi_{rr}(z) &= H(z)H^*(1/z^*) + N_0 \\ \Phi_{Ir}(z) &= H^*(1/z^*) \end{aligned}$$

This results in

$$F(z) = \frac{H^*(1/z^*)}{H(z)H^*(1/z^*) + N_0} B(z).$$

Recall that

$$F_{\text{LE}}(z) = \frac{H^*(1/z^*)}{H(z)H^*(1/z^*) + N_0}$$

is the optimum filter for linear MMSE equalization. This means the optimum FFF for MMSE–DFE is the cascade of a optimum linear equalizer and the FBF $B(z)$.

- **FBF Optimization**

The Z–transform $E(z)$ of the error signal $e[k]$ is given by

$$E(z) = F(z)Z(z) + (F(z)H(z) - B(z))I(z).$$

Adopting the optimum $F(z)$, we obtain for the $\mathcal{Z}$–transform of the autocorrelation sequence

$$\begin{aligned}\Phi_{ee}(z) &= \mathcal{E}\{E(z)E^*(1/z^*)\}\\ &= \frac{B(z)B^*(1/z^*)H(z)H^*(1/z^*)}{(H(z)H^*(1/z^*)+N_0)^2}N_0\\ &\quad+\frac{N_0^2B(z)B^*(1/z^*)}{(H(z)H^*(1/z^*)+N_0)^2}\\ &= B(z)B^*(1/z^*)\frac{H(z)H^*(1/z^*)+N_0}{(H(z)H^*(1/z^*)+N_0)^2}N_0\\ &= B(z)B^*(1/z^*)\underbrace{\frac{N_0}{H(z)H^*(1/z^*)+N_0}}_{\Phi_{e_le_l}(z)},\end{aligned}$$

where $e_l[k]$ denotes the error signal at the output of the optimum linear MMSE equalizer, and $\Phi_{e_le_l}(z)$ is the $\mathcal{Z}$–transform of the autocorrelation sequence of $e_l[k]$.

The optimum FBF filter will minimize the variance of $e_l[k]$. Therefore, the optimum prediction–error filter for $e_l[k]$ is the optimum filter $B(z)$. Consequently, the optimum FBF can be defined as

$$B(z) = \frac{1}{Q(z)},$$

where $Q(z)$ is obtained by *spectral factorization* of $\Phi_{e_le_l}(z)$

$$\begin{aligned}\Phi_{e_le_l}(z) &= \frac{N_0}{H(z)H^*(1/z^*)+N_0}\\ &= \sigma_e^2 Q(z)Q^*(1/z^*).\end{aligned}$$

The coefficients of $q[k]$, $k \geq 1$, can be calculated recursively as

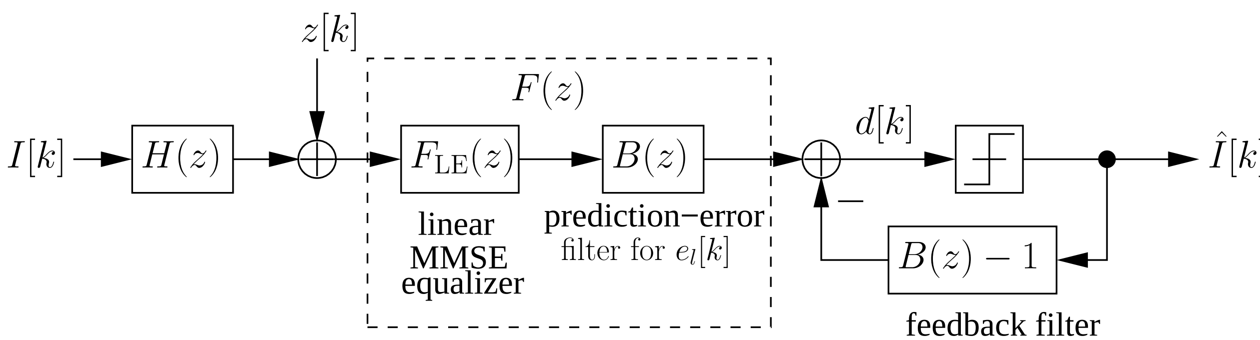
$$q[k] = \sum_{\mu=0}^{k-1}\frac{k-\mu}{k}q[\mu]\beta[k-\mu], \quad k \geq 1$$

with

$$\beta[\mu] = T \int_{-1/(2T)}^{1/(2T)} \ln\left[\Phi_{e_l e_l}(e^{j2\pi fT})\right] e^{j2\pi\mu fT}\, \mathrm{d}f.$$

The error variance σ_e^2 is given by

$$\sigma_e^2 = \exp\left(T \int_{-1/(2T)}^{1/(2T)} \ln\left[\frac{N_0}{|H(e^{j2\pi fT})|^2 + N_0}\right] \mathrm{d}f \right).$$



■ **Overall Channel**

The overall forward transfer function is given by

$$\begin{aligned} H_{\mathrm{ov}}(z) &= F(z)H(z) \\ &= \frac{H(z)H^*(1/z^*)}{H(z)H^*(1/z^*) + N_0} B(z) \\ &= \left(1 - \frac{N_0}{H(z)H^*(1/z^*) + N_0}\right) B(z) \\ &= B(z) - \frac{\sigma_e^2}{B^*(1/z^*)}. \end{aligned}$$

Therefore, the bias $h_{\rm ov}[0]$ is given by

$$h_{\rm ov}[0] = 1 - \sigma_e^2.$$

The part of the transfer function that characterizes the *pre–cursors* is given by

$$-\sigma_e^2 \left(\frac{1}{B^*(1/z^*)} - 1 \right),$$

whereas the part of the transfer function that characterizes the *post–cursors* is given by

$$B(z) - 1$$

Hence, the error signal is composed of bias, pre–cursor ISI, and noise. The post–cursor ISI is perfectly canceled by the FBF.

- **SNR**

  Taking into account the bias, it can be shown that the SNR for optimum MMSE DFE is given by

$$\begin{aligned} \mathrm{SNR}_{\mathrm{MMSE-DFE}} &= \frac{1}{\sigma_e^2} - 1 \\ &= \exp\left(T \int\limits_{-1/(2T)}^{1/(2T)} \ln\left[\frac{|H(e^{j2\pi fT})|^2}{N_0} + 1 \right] \mathrm{d}f \right) - 1. \end{aligned}$$

- **Remark**

  For high SNRs, ZF–DFE and MMSE–DFE become equivalent. In that case, the noise variance is comparatively small and also the MMSE criterion leads to a complete elimination of the ISI.

6.5 MMSE–DFE with FIR Filters

- Since IIR filters cannot be realized, in practice FIR filters have to be employed.
- **Error Signal** $e[k]$
  If we denote FFF length and order by L_F and $q_F = L_F - 1$, respectively, and the FBF length and order by L_B and $q_B = L_B - 1$, respectively, we can write the slicer input signal as

$$d[k] = \sum_{\kappa=0}^{q_F} f[\kappa] r_b[k-\kappa] - \sum_{\kappa=1}^{q_B} b[\kappa] I[k-k_0-\kappa],$$

  where we again allow for a decision delay k_0, $k_0 \geq 0$.

Using vector notation, the error signal can be expressed as

$$\begin{aligned} e[k] &= d[k] - I[k-k_0] \\ &= \boldsymbol{f}^H \boldsymbol{r}_b[k] - \boldsymbol{b}^H \boldsymbol{I}[k-k_0-1] - I[k-k_0] \end{aligned}$$

with

$$\begin{aligned} \boldsymbol{f} &= [f[0]\ \ f[1]\ \ \ldots\ \ f[q_F]]^H \\ \boldsymbol{b} &= [b[1]\ \ f[2]\ \ \ldots\ \ b[q_B]]^H \\ \boldsymbol{r}_b[k] &= [r_b[k]\ \ r_b[k-1]\ \ \ldots\ \ r_b[k-q_F]]^T \\ \boldsymbol{I}[k-k_0-1] &= [I[k-k_0-1]\ \ I[k-k_0-2]\ \ \ldots\ \ I[k-k_0-q_B]]^T. \end{aligned}$$

- **Error Variance** J

  The error variance can be obtained as

$$\begin{aligned} J &= \mathcal{E}\{|e[k]|^2\} \\ &= \boldsymbol{f}^H \mathcal{E}\{\boldsymbol{r}_b[k]\boldsymbol{r}_b^H[k]\}\boldsymbol{f} + \boldsymbol{b}^H \mathcal{E}\{\boldsymbol{I}[k-k_0-1]\boldsymbol{I}^H[k-k_0-1]\}\boldsymbol{b} \\ &\quad -\boldsymbol{f}^H \mathcal{E}\{\boldsymbol{r}_b[k]\boldsymbol{I}^H[k-k_0-1]\}\boldsymbol{b} - \boldsymbol{b}^H \mathcal{E}\{\boldsymbol{I}[k-k_0-1]\boldsymbol{r}_b^H[k]\}\boldsymbol{f} \\ &\quad -\boldsymbol{f}^H \mathcal{E}\{\boldsymbol{r}_b[k]I^*[k-k_0]\} - \mathcal{E}\{\boldsymbol{r}_b^H[k]I[k-k_0]\}\boldsymbol{f} \\ &\quad +\boldsymbol{b}^H \mathcal{E}\{\boldsymbol{I}[k-k_0-1]I^*[k-k_0]\} + \mathcal{E}\{\boldsymbol{I}^H[k-k_0-1]I[k-k_0]\}\boldsymbol{b}. \end{aligned}$$

  Since $I[k]$ is an i.i.d. sequence and the noise $z[k]$ is white, the following identities can be established:

$$\begin{aligned}
\mathcal{E}\{\boldsymbol{I}[k-k_0-1]I^*[k-k_0]\} &= \boldsymbol{0} \\
\mathcal{E}\{\boldsymbol{I}[k-k_0-1]\boldsymbol{I}^H[k-k_0-1]\} &= \boldsymbol{I} \\
\mathcal{E}\{\boldsymbol{r}_b[k]\boldsymbol{I}^H[k-k_0-1]\} &= \boldsymbol{H} \\
\boldsymbol{H} = \begin{bmatrix} h[k_0+1] & \dots & h[k_0+q_B] \\ h[k_0] & \dots & h[k_0+q_B-1] \\ \vdots & \ddots & \vdots \\ h[k_0+1-q_F] & \dots & h[k_0+q_B-q_F] \end{bmatrix} & \\
\mathcal{E}\{\boldsymbol{r}_b[k]I^*[k-k_0]\} &= \boldsymbol{h} \\
\boldsymbol{h} &= [h[k_0] \;\; h[k_0-1] \;\; \dots \;\; h[k_0-q_F]]^T \\
\mathcal{E}\{\boldsymbol{r}_b[k]\boldsymbol{r}_b^H[k]\} &= \boldsymbol{\Phi}_{hh} + \sigma_n^2\boldsymbol{I},
\end{aligned}$$

where $\boldsymbol{\Phi}_{hh}$ denotes the channel autocorrelation matrix. With these definitions, we get

$$\begin{aligned}
J &= \boldsymbol{f}^H(\boldsymbol{\Phi}_{hh} + \sigma_n^2\boldsymbol{I})\boldsymbol{f} + \boldsymbol{b}^H\boldsymbol{b} + 1 \\
&\quad -\boldsymbol{f}^H\boldsymbol{H}\boldsymbol{b} - \boldsymbol{b}^H\boldsymbol{H}^H\boldsymbol{f} - \boldsymbol{f}^H\boldsymbol{h} - \boldsymbol{h}^H\boldsymbol{f}
\end{aligned}$$

- **Optimum Filters**

  The optimum filter settings can be obtained by differentiating J with respect to $\boldsymbol{f}^*$ and $\boldsymbol{b}^*$, respectively.

$$\begin{aligned}
\frac{\partial J}{\partial \boldsymbol{f}^*} &= (\boldsymbol{\Phi}_{hh} + \sigma_n^2\boldsymbol{I})\boldsymbol{f} - \boldsymbol{H}\boldsymbol{b} - \boldsymbol{h} \\
\frac{\partial J}{\partial \boldsymbol{b}^*} &= \boldsymbol{b} - \boldsymbol{H}^H\boldsymbol{f}
\end{aligned}$$

  Setting the above equations equal to zero and solving for $\boldsymbol{f}$, we get

$$\boldsymbol{f}_{\mathrm{opt}} = \left(\left(\boldsymbol{\Phi}_{hh} - \boldsymbol{H}\boldsymbol{H}^H\right) + \sigma_n^2\boldsymbol{I}\right)^{-1}\boldsymbol{h}$$

The optimum FBF is given by

$$\boldsymbol{b}_{\text{opt}} = \boldsymbol{H}^H \boldsymbol{f} = [h_{\text{ov}}[k_0+1] \;\; h_{\text{ov}}[k_0+1] \;\; \ldots \;\; h_{\text{ov}}[k_0+q_B]]^H,$$

where $h_{\text{ov}}[k]$ denotes the overall impulse response comprising channel and FFF. This means the FBF cancels perfectly the postcursor ISI.

- **MMSE**
  The MMSE is given by

$$\begin{aligned} J_{\min} &= 1 - \boldsymbol{h}^H \left(\left(\boldsymbol{\Phi}_{hh} - \boldsymbol{H}\boldsymbol{H}^H \right) + \sigma_n^2 \boldsymbol{I} \right)^{-1} \boldsymbol{h} \\ &= 1 - \boldsymbol{f}_{\text{opt}}^H \boldsymbol{h}. \end{aligned}$$

- **Bias**
  The bias is given by

$$h[k_0] = \boldsymbol{f}_{\text{opt}}^H \boldsymbol{h} = 1 - J_{\min} < 1.$$