In-class examples, Oct 14

① $\dot{x} = \begin{bmatrix} -1 & 1 \\ 0 & -2 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u, \quad x(0) = \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \quad u(t) = s(t)$ (step input)

$$x(t) = e^{At} x(0) + \int_0^{\infty} e^{A(t-\tau)} B u(\tau) d\tau$$
$$= \mathcal{L}^{-1} \{ (sI-A)^{-1} x(0) + (sI-A)^{-1} B U(s) \}$$

$$(sI-A)^{-1} = \begin{bmatrix} s+1 & -1 \\ 0 & s+2 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{1}{s+1} & \frac{1}{(s+1)(s+2)} \\ 0 & \frac{1}{s+2} \end{bmatrix}$$

Since $\mathcal{L}\{e^{-at}\} = \int_0^{\infty} e^{-at} e^{-st} dt = \frac{-1}{s+a} e^{-(s+a)t} \Big|_0^{\infty} = \frac{1}{s+a}$,

and $\mathcal{L}\{s(t)\} = \int_0^{\infty} e^{-st} dt \cdot 1 = -\frac{1}{s} e^{-st} \Big|_0^{\infty} = +\frac{1}{s}$,

$$\mathcal{L}^{-1}\{(sI-A)^{-1} x(0)\} = \mathcal{L}^{-1}\left\{ \begin{bmatrix} \frac{-1}{s+1} \\ 0 \end{bmatrix} \right\} = \begin{bmatrix} -e^{-t} \\ 0 \end{bmatrix}$$

$$\mathcal{L}^{-1}\{(sI-A)^{-1} B U(s)\} = \mathcal{L}^{-1}\left\{ \begin{bmatrix} \frac{1}{(s+1)(s+2)} \\ \frac{1}{s+2} \end{bmatrix} \cdot \frac{1}{s} \right\}$$

Use partial fraction expansion

$$\frac{1}{s(s+1)(s+2)} = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s+2}$$

$$1 = A(s^2 + 3s + 2) + B(s^2 + 2s) + C(s^2 + s)$$

$$0 \cdot s^2 + 0 \cdot s + 1 = s^2(A+B+C) + s(3A + 2B + C) + (2A)$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 3 & 2 & 1 \\ 2 & 0 & 0 \end{bmatrix} \begin{bmatrix} A \\ B \\ C \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \Rightarrow \begin{bmatrix} A \\ B \\ C \end{bmatrix} = \begin{bmatrix} 1/2 \\ -1 \\ 1/2 \end{bmatrix}$$

and $\frac{1}{s(s+2)} = \frac{D}{s} + \frac{E}{s+2}$

$$1 = D(s+2) + sE$$

$$0 \cdot s + 1 = s(D+E) + 2D \Rightarrow \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} D \\ E \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \Rightarrow \begin{bmatrix} D \\ E \end{bmatrix} = \begin{bmatrix} 1/2 \\ -1/2 \end{bmatrix}$$

$$\therefore \mathcal{L}^{-1}\{(sI-A)^{-1}BU(s)\} = \mathcal{L}^{-1}\left\{\begin{bmatrix} \frac{1/2}{s} - \frac{1}{s+1} + \frac{1/2}{s+2} \\ \frac{1/2}{s} - \frac{1/2}{s+2} \end{bmatrix}\right\} = \begin{bmatrix} 1/2\, s(t) - e^{-t} + 1/2\, e^{-2t} \\ 1/2\, s(t) - 1/2\, e^{-2t} \end{bmatrix}$$

So $x(t) = \begin{bmatrix} \frac{1}{2} - 2e^{-t} + \frac{1}{2}e^{-2t} \\ \frac{1}{2} - \frac{1}{2}e^{-2t} \end{bmatrix}$ for $t \geq 0$.

(2) $\dot{x} = \begin{bmatrix} 0 & -1 \\ 1 & -2 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$. Find e^{At}.

Use diagonalization / Jordan form:

$$0 = |\lambda I - A| = \begin{vmatrix} \lambda & 1 \\ -1 & \lambda+2 \end{vmatrix} = \lambda^2 + 2\lambda + 1 = (\lambda+1)^2 \Rightarrow \text{repeated eigval } \lambda = -1$$

$$(\lambda I - A)v = \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} v_{11} \\ v_{12} \end{bmatrix} = 0 \Rightarrow v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Generalized eigvector:

$$(\lambda I - A)v_2 = v_1$$

$$\begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} v_{21} \\ v_{22} \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \Rightarrow v_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\therefore A = TJT^{-1}, \quad T = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}, \quad J = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}, \quad T^{-1} = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$$

So
$$e^{At} = Te^{Jt}T^{-1} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} e^{-t} & te^{-t} \\ 0 & e^{-t} \end{bmatrix}\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} e^{-t} & te^{-t} + e^{-t} \\ e^{-t} & te^{-t} + 2e^{-t} \end{bmatrix}\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} -te^{-t} + e^{-t} & te^{-t} \\ -te^{-t} & te^{-t} + e^{-t} \end{bmatrix} = e^{-t}\begin{bmatrix} 1-t & t \\ -t & 1+t \end{bmatrix}$$