# EECE 568: Control Systems
# Handout #1: Solutions to Linear Differential and Difference Equations


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September 28, 2009


## Continuous-Time Systems

### Time-Varying

Consider the multi-input, multi-output, linear time-varying system

$$\begin{array}{rcll} \dot{x}(t) & = & A(t)x(t) + B(t)u(t), & x \in \mathbb{R}^n,\ u \in \mathbb{R}^m,\ \text{with } x(t_0) = x_0 \\ y(t) & = & C(t)x(t) + D(t)u(t), & y \in \mathbb{R}^p \end{array} \tag{1}$$

with known initial condition $x_0$ and known input signal $u(t)$, $t \in \mathbb{R}$. Assume the system is causal and that $A(t), B(t), C(t), D(t)$ are continuous matrices of the appropriate dimensions.
The *state transition matrix*, determined by the method of successive approximations,

$$\Phi(t, t_0) = \lim_{m \to \infty} \phi_m(t), \quad \phi_m(t) = x_0 + \int_{t_0}^{t} A(s)\phi_{m-1}(s)ds,\ m \in R_+,\ \ \phi_0(t) = x_0 \tag{2}$$

is required to determine the unique *solution* to the differential equation.

$$x(t) = \Phi(t, t_0)x_0 + \int_{t_0}^{t} \Phi(t, s)B(s)u(s)ds, \tag{3}$$

Notice that the solution can be broken up into two components: the *zero-input response*, which occurs when the input is $u(t) = 0$, and the *zero-state response*, which occurs when the initial condition is $x_0 = 0$. One the solution is known, the output signal can be calculated.

$$y(t) = C(t)\Phi(t, t_0)x_0 + C(t)\int_{t_0}^{t} \Phi(t, s)B(s)u(s)ds + D(t)u(t) \tag{4}$$

### Time-Invariant

Consider the multi-input, multi-output, linear time-invariant system

$$\begin{array}{rcll} \dot{x}(t) & = & Ax(t) + Bu(t), & x \in \mathbb{R}^n,\ u \in \mathbb{R}^m,\ \text{with } x(t_0) = x_0 \\ y(t) & = & Cx(t) + Du(t), & y \in \mathbb{R}^p \end{array} \tag{5}$$

with known initial condition $x_0$ and known input signal $u(t)$, $t \in \mathbb{R}$. Assume the system is causal and that $A, B, C, D$ are constant-valued matrices of the appropriate dimensions.
The state transition matrix

$$\Phi(t, t_0) = e^{A(t-t_0)} = \sum_{k=0}^{\infty} A^k \frac{(t-t_0)^k}{k!} \tag{6}$$

is the matrix exponential, often written as $\Phi(t - t_0)$. The solution to the differential equation is

$$x(t) = \Phi(t-t_0)x_0 + \int_{t_0}^{t} \Phi(t-s)Bu(s)ds, \tag{7}$$

One the solution is known, the output signal can be calculated.

$$y(t) = C\Phi(t-t_0)x_0 + C\int_{t_0}^{t} \Phi(t-s)Bu(s)ds + Du(t) \tag{8}$$

### Properties of the State-Transition Matrix

$$\begin{aligned} \Phi(t,t) &= I \\ \dot{\Phi}(t,t_0) &= A(t)\Phi(t,t_0) \\ \Phi(t,t_0) &= \Phi(t,t_1)\Phi(t_1,t_0) \\ \Phi^{-1}(t,t_0) &= \Phi(t_0,t) \end{aligned} \tag{9}$$

In addition, some properties of the matrix exponential are:

$$\begin{aligned} e^{At_1}e^{At_2} &= e^{A(t_1+t_2)} \\ Ae^{At} &= e^{At}A \\ \left(e^{At}\right)^{-1} &= e^{-At} \end{aligned} \tag{10}$$

## Discrete-Time Systems

### Time-Varying

For the multi-input, multi-output, linear time-varying system

$$\begin{aligned} x[k+1] &= A(k)x[k] + B(k)u[k], \qquad x \in \mathbb{R}^n,\ u \in \mathbb{R}^m,\ \text{with } x(k_0) = x_0 \\ y[k] &= C(k)x[k] + D(k)u[k], \qquad y \in \mathbb{R}^p \end{aligned} \tag{11}$$

with known initial condition $x_0$ and known input sequence $u[k]$, $k \in \mathbb{Z}$. Assume the system is causal and that $A(k), B(k), C(k), D(k)$ are continuous matrices of the appropriate dimensions.
The *state transition matrix*

$$\Phi(k, k_0) = \prod_{j=k_0}^{k-1} A(j),\ k \geq k_0 \tag{12}$$

determines the *solution* to the difference equation for $k > k_0$.

$$x[k] = \Phi(k, k_0)x_0 + \sum_{j=k_0}^{k-1} \Phi(k, j+1)B(j)u[j], \tag{13}$$

The solution can again be broken up into the *zero-input response* and the *zero-state response*. One the solution is known, the output sequence can be calculated.

$$y[k] = C(k)\Phi(k,k_0)x_0 + C(k)\sum_{j=k_0}^{k-1}\Phi(k,j+1)B(j)u[j] + D(k)u[k],\ k \geq k_0 \tag{14}$$

**Time-Invariant**

For the multi-input, multi-output, linear time-invariant system

$$\begin{aligned} x[k+1] &= A(k)x[k] + B(k)u[k], \qquad x \in \mathbb{R}^n,\ u \in \mathbb{R}^m,\ \text{with } x(k_0) = x_0 \\ y[k] &= C(k)x[k] + D(k)u[k], \qquad y \in \mathbb{R}^p \end{aligned} \tag{15}$$

with known initial condition $x_0$ and known input sequence $u[k]$, $k \in \mathbb{Z}$. Assume the system is causal and that $A, B, C, D$ are constant-valued matrices of the appropriate dimensions.
With the *state transition matrix*

$$\Phi(k,k_0) = A^{k-k_0},\ k \geq k_0 \tag{16}$$

the *solution* to the difference equation for $k > k_0$ is given by

$$x[k] = A^{k-k_0}x_0 + \sum_{j=k_0}^{k-1} A^{k-(j+1)}Bu[j] \tag{17}$$

The solution can again be broken up into the *zero-input response* and the *zero-state response*. One the solution is known, the output sequence can be calculated.

$$y[k] = CA^{k-k_0}x_0 + C\sum_{j=k_0}^{k-1} A^{k-(j+1)}Bu[j] + Du[k],\ k \geq k_0 \tag{18}$$

**Properties of the State-Transition Matrix**

$$\begin{aligned} \Phi(k,k) &= I, & k \geq k_0 \\ \Phi(k+1,k_0) &= A(k)\Phi(k,k_0), & k \geq k_0 \\ \Phi(k,k_0) &= \Phi(k,k_1)\Phi(k_1,k_0), & k \geq k_1 \geq k_0 \end{aligned} \tag{19}$$

Unlike the continuous-time case, the state-transition matrix is not always invertible since the system matrix $A(k)$ or $A$ may be singular.