EECE 568: Control Systems
Handout #2: State transition matrix for LTI systems

Dr. Meeko Oishi
http://courses.ece.ubc.ca/568

October 15, 2009

Similarity Transformations

For diagonalizable A, every eigenvalue λ_i solves $A\lambda_i = v_i\lambda_i$ with eigenvector v_i. With n independent eigenvectors as the columns of the $n \times n$ matrix V, and the eigenvalues $\lambda_1, \cdots, \lambda_n$ the elements of the diagonal matrix Λ,

$$A = V\Lambda V^{-1}, \quad \Lambda = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} \tag{1}$$

For non-diagonalizable (defective) A, generalized eigenvectors v_j^i corresponding to eigenvalue λ_i solve $Av_{j+1}^i = \lambda_i v_j^i$, $j \in \{1, \cdots, n_i - 1\}$. The generalized eigenvectors form the columns of the matrix T. The block-diagonal Jordan matrix J consists of bidiagonal Jordan blocks J_i, $i \in \{1, \cdots, q\}$, with $\sum_{i=1}^{q} n_i = n$.

$$A = TJT^{-1}, \quad J = \begin{bmatrix} J_1 & & & \\ & J_2 & & \\ & & \ddots & \\ & & & J_q \end{bmatrix}, \text{ with } J_i = \begin{bmatrix} \lambda_i & 1 & \cdots & 0 \\ 0 & \lambda_i & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 \\ 0 & \cdots & 0 & \lambda_i \end{bmatrix} \in \mathbb{R}^{n_i \times n_i} \tag{2}$$

Continuous-Time Systems

For the multi-input, multi-output, linear time-invariant system

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), \qquad x \in \mathbb{R}^n,\ u \in \mathbb{R}^m, \text{ with } x(t_0) = x_0 \\ y(t) &= Cx(t) + Du(t), \qquad y \in \mathbb{R}^p \end{aligned} \tag{3}$$

with real-valued matrices A, B, C, D of the appropriate dimensions, the state transition matrix $\Phi(t, t_0) = e^{A(t-t_0)} = \sum_{k=0}^{\infty} A^k \frac{(t-t_0)^k}{k!}$ is the matrix exponential. Assume $t_0 = 0$ for convenience in the following.

For diagonalizable A, the state transition matrix is $e^{At} = Ve^{\Lambda t}V^{-1}$, with

$$e^{\Lambda t} = \begin{bmatrix} e^{\lambda_1 t} & 0 & \cdots & 0 \\ 0 & e^{\lambda_2 t} & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & e^{\lambda_n t} \end{bmatrix} \tag{4}$$

and the resolvent is $(sI - A)^{-1} = V(sI - \Lambda)^{-1}V^{-1}$, with

$$(sI - \Lambda)^{-1} = \begin{bmatrix} \frac{1}{s+\lambda_1} & 0 & \cdots & 0 \\ 0 & \frac{1}{s+\lambda_2} & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{1}{s+\lambda_n} \end{bmatrix} \tag{5}$$

For defective A, the state-transition matrix is $e^{At} = Te^{Jt}T^{-1}$, with

$$e^{Jt} = \begin{bmatrix} e^{J_1 t} & 0 & \cdots & 0 \\ 0 & e^{J_2 t} & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & e^{J_n t} \end{bmatrix}, \text{ and } e^{J_i t} = \begin{bmatrix} e^{\lambda_i t} & te^{\lambda_i t} & \cdots & \frac{t^{n_i}}{n_i!}e^{\lambda_i t} \\ 0 & e^{\lambda_i t} & \ddots & \vdots \\ \vdots & \ddots & \ddots & te^{\lambda_i t} \\ 0 & \cdots & 0 & e^{\lambda_i t} \end{bmatrix} \in \mathbb{R}^{n_i \times n_i} \tag{6}$$

and the resolvent is $(sI - A)^{-1} = V(sI - \Lambda)^{-1}V^{-1}$, with

$$\begin{aligned} (sI - J)^{-1} &= \begin{bmatrix} (sI - J_1)^{-1} & 0 & \cdots & 0 \\ 0 & (sI - J_2)^{-1} & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & (sI - J_n)^{-1} \end{bmatrix}, \text{ and} \\ (sI - J_i)^{-1} &= \begin{bmatrix} \frac{1}{s+\lambda_i} & \frac{1}{(s+\lambda_i)^2} & \cdots & \frac{1}{(s+\lambda_i)^n_i} \\ 0 & \frac{1}{s+\lambda_i} & \ddots & \vdots \\ \vdots & \ddots & \ddots & \frac{1}{(s+\lambda_i)^2} \\ 0 & \cdots & 0 & \frac{1}{s+\lambda_i} \end{bmatrix} \in \mathbb{R}^{n_i \times n_i} \end{aligned} \tag{7}$$

Discrete-Time Systems

For the multi-input, multi-output, linear time-invariant system

$$\begin{aligned} x[k+1] &= Ax[k] + Bu[k], \qquad x \in \mathbb{R}^n,\ u \in \mathbb{R}^m,\ \text{with } x(k_0) = x_0 \\ y[k] &= Cx[k] + Du[k], \qquad y \in \mathbb{R}^p \end{aligned} \tag{8}$$

with real-valued matrices A, B, C, D of the appropriate dimensions, the state transition matrix is $\Phi(k, k_0) = A^{(k-k0)}$. Assume $k_0 = 0$ for convenience in the following.

For diagonalizable A, the state transition matrix is $A^k = V\Lambda^k V^{-1}$, with

$$\Lambda^k = \begin{bmatrix} (\lambda_1)^k & 0 & \cdots & 0 \\ 0 & (\lambda_2)^k & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & (\lambda_n)^k \end{bmatrix} \tag{9}$$

For defective A, the state transition matrix is $A^k = TJ^kT^{-1}$, with

$$J^k = \begin{bmatrix} (J_1)^k & 0 & \cdots & 0 \\ 0 & (J_2)^k & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & (J_n)^k \end{bmatrix}, \text{ and } (J_i)^k = \begin{bmatrix} (\lambda_i)^k & k(\lambda_i)^{k-1} & \cdots & \frac{k!}{1!(k-1)!}(\lambda_i)^1 \\ 0 & (\lambda_i)^k & \ddots & \vdots \\ \vdots & \ddots & \ddots & k(\lambda_i)^{k-1} \\ 0 & \cdots & 0 & (\lambda_i)^k \end{bmatrix} \in \mathbb{R}^{n_i \times n_i} \tag{10}$$

The $(j+1)$th column in the first row of $(J_i)^k$ has the value $\frac{k!}{(k-j)!j!}(\lambda_i)^{k-j}$.

The resolvent (used in calculation of transfer functions in the z-domain) is the same as was calculated for the continuous-time system, for A diagonalizable and for A defective, respectively.