$\boxed{1}$ 

<u>Linearity</u>:

For $\left.\begin{matrix} x_1(t_0) \\ u_1(t) \end{matrix}\right\} \to y_1(t)$ and $\left.\begin{matrix} x_2(t_0) \\ u_2(t) \end{matrix}\right\} \to y_2(t)$,

$$\left.\begin{matrix} \alpha_1 x_1(t_0) + \alpha_2 x_2(t_0) \\ \alpha u_1(t) + \alpha_2 u_2(t) \end{matrix}\right\} \to y_3(t)$$

We know $\left.\begin{matrix} x(t_0) \\ u_1(t) \end{matrix}\right\} \to y_1(t)$, $\left.\begin{matrix} x(t_0) \\ u_2(t) \end{matrix}\right\} \to y_2(t)$, $\left.\begin{matrix} x(t_0) \\ u_3(t) \end{matrix}\right\} \to y_3(t)$.

1\. A: $\alpha_1 = 1, \alpha_2 = 1$: $\left.\begin{matrix} x(t_0) + x(t_0) \\ u_1(t) + u_2(t) \end{matrix}\right\} = \left.\begin{matrix} 2x(t_0) \\ u_1(t) + u_2(t) \end{matrix}\right\} \not\to y_1(t) + y_2(t)$

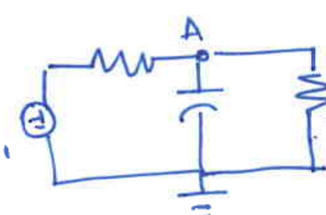
B: $\alpha_1 = \frac{1}{2}, \alpha_2 = \frac{1}{2}$: $\left.\begin{matrix} \frac{1}{2}x(t_0) + \frac{1}{2}x(t_0) \\ \frac{1}{2}u_1(t) + \frac{1}{2}u_2(t) \end{matrix}\right\} = \left.\begin{matrix} x(t_0) \\ \frac{1}{2}(u_1(t)) + \frac{1}{2}u_2(t) \end{matrix}\right\} \to y_1(t)\cdot\frac{1}{2} + y_2(t)\cdot\frac{1}{2}$ ✓.

C: $\alpha_1 = 1, \alpha_2 = -1$: $\left.\begin{matrix} x(t_0) - x(t_0) \\ u_1(t) - u_2(t) \end{matrix}\right\} = \left.\begin{matrix} 0 \\ u_1(t) - u_2(t) \end{matrix}\right\} \not\to y_1(t) - y_2(t)$

So $\boxed{B}$ is true.

2\. $\boxed{A, B, C}$ are true. (zero-state response)

$\boxed{2}$ KCL at node A:




$$\frac{v_1 - v_R}{R} = i_R + C\frac{dv_R}{dt}$$

$$\frac{dv_R}{dt} = -\frac{v_R}{RC} - \frac{1.5}{C}v_R^3 + \frac{v_1}{RC}$$

1\. With $x = v_R$, $u = v_1$

$$\dot{x} = -\frac{1}{RC}x - \frac{1.5}{C}x^3 + \frac{1}{RC}u = -x - 1.5x^3 + u$$

2\. For $u_0(t) = 14$, $\dot{x} = -x(1 + 1.5x^2) + u_0(t) = 0$ in steady-state

$$x(1 + 1.5x^2) = 14$$

$$x_0(t) = 2$$

(Solved using Matlab's 'roots' function)

For $\delta x(t) \triangleq x(t) - x_0(t)$, $\delta u(t) = u(t) - u_0(t)$

$$\left.\frac{\partial f}{\partial x}\right|_{x_0(t), u_0(t)} = -4.5x^2 - 1 \Big|_{x_0(t), u_0(t)} = -19$$

$$\left.\frac{\partial f}{\partial u}\right|_{x_0(t), u_0(t)} = 1 \Big|_{x_0(t), u_0(t)} = 1.$$

Linearized dynamics $\dot{\delta x} = -19\delta x + \delta u$

output: $y(t) = x \implies \delta y(t) = \delta x(t)$.

3. Sol'n to linearized dynamics with $\delta u(t) = u(t) - u_0(t) = 15 - 14 = 1$ for $t \geq 0$.

$$\delta x(t) = e^{-19t}\delta x(0) + \int_0^t e^{-19(t-\tau)}\, d\tau$$

$$= 0 + e^{-19t} \frac{1}{19} e^{19\tau}\Big|_0^t$$

$$= \frac{1}{19}(1 - e^{-19t})$$

See attached plot.

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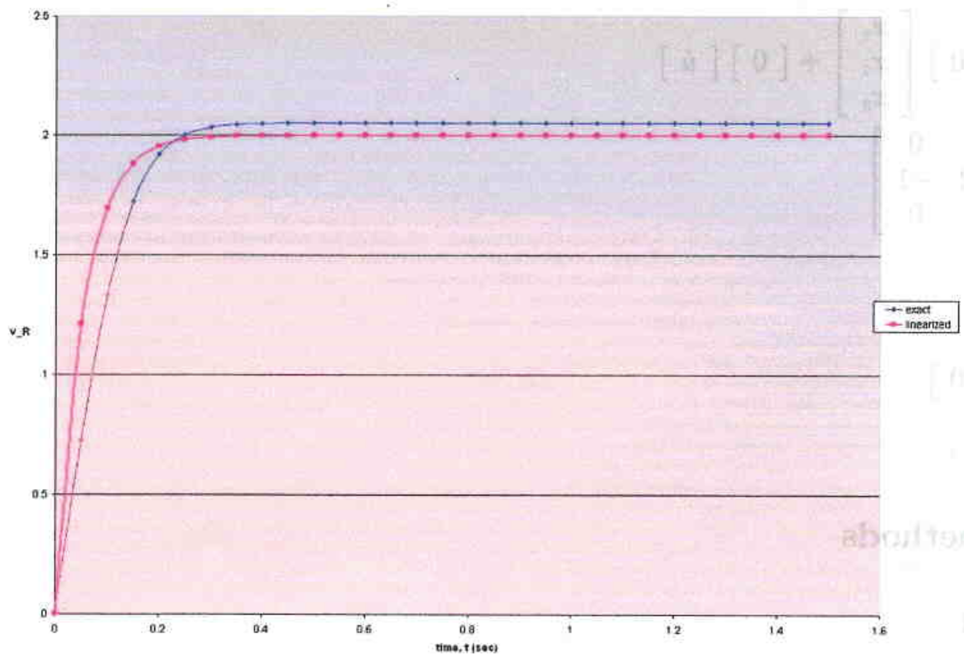
$\boxed{3}$ 1. Choose $x = \begin{bmatrix} y \\ \dot{y} \\ u \end{bmatrix} \in \mathbb{R}^3$, input $v = \dot{u}$, output $y$

Then $\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ +3 & -2 & -1 \\ 0 & 0 & 1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} v \qquad = Ax + Bv$

$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} x + 0 \cdot v \qquad = Cx + Dv$

2. $Y(s) = C(sI - A)^{-1} B U(s)$

$$\underbrace{\frac{Y(s)}{U(s)}}_{G(s)} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} s & -1 & 0 \\ -3 & s+2 & 1 \\ 0 & 0 & s-1 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$




Graph from Andrew Larkin.

$$= [1 \quad 0 \quad 0] \; \frac{1}{s(s+2)(s-1)} \begin{bmatrix} (s+2)(s-1) & 3(s-1) & 0 \\ (s-1) & s(s-1) & 0 \\ -1 & s & s(s+2)-3 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$= \frac{3(s-1)}{s(s+2)(s-1)} = \frac{3}{s(s+2)}$$

$\boxed{4}$ $\dot{x} = 3x, \quad x(0) = 5$

1. $x[i+1] = x[i] + h \cdot 3x[i] = x[i](1+3h)$

$$x[1] = 5(1+3h)$$
$$x[2] = 5(1+3h)^2$$
$$x[3] = 5(1+3h)^3$$
$$\vdots$$
$$x[k] = 5(1+3h)^k = 5A_E^k, \quad A_E \triangleq 1+3h$$

2. $x[i+1] = x[i] + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4),$

$$k_1 = h \cdot 3x[i]$$
$$k_2 = h \cdot 3(x[i] + \tfrac{1}{2}k_1) = h \cdot 3x[i] + h \cdot \tfrac{3}{2} \cdot h \cdot 3x[i] = 3hx[i](1 + \tfrac{3h}{2})$$
$$k_3 = h \cdot 3(x[i] + \tfrac{1}{2}k_2) = 3hx[i] + \tfrac{3}{2}h \cdot 3hx[i](1 + \tfrac{3h}{2})$$
$$= 3hx[i](1 + \tfrac{3}{2}h(1 + \tfrac{3h}{2}))$$
$$= 3hx[i](1 + \tfrac{3}{2}h + (\tfrac{3h}{2})^2)$$
$$k_4 = h \cdot 3(x[i] + k_3) = 3hx[i] + 3h \cdot 3hx[i](1 + \tfrac{3}{2}h + (\tfrac{3h}{2})^2)$$
$$= 3hx[i](1 + 3h(1 + \tfrac{3}{2}h + (\tfrac{3}{2}h)^2))$$

$$\therefore k = \frac{1}{6} \cdot 3hx[i]\left(1 + 2(1 + \tfrac{3h}{2}) + 2(1 + \tfrac{3}{2}h + (\tfrac{3h}{2})^2) + (1 + 3h(1 + \tfrac{3}{2}h + (\tfrac{3h}{2})^2))\right)$$
$$= \frac{1}{2}hx[i]\left(1 + 2 + 3h + 2 + 3h + \frac{(3h)^2}{2} + 1 + 3h + \frac{3^2h^2}{2} + \frac{(3h)^3}{4}\right)$$
$$= \frac{1}{2}hx[i]\left(6 + 9h + (3h)^2 + \frac{(3h)^3}{4}\right)$$
$$= \frac{3}{2}hx[i]\left(2 + 3h + 3h^2 + \frac{9h^3}{4}\right)$$

$$\therefore \quad x[i+1] = x[i] + \frac{3}{2} h\, x[i] \left(2 + 3h + 3h^2 + \frac{9}{4} h^3\right)$$

$$= x[i]\left(1 + \frac{3}{2} h \left(2 + 3h + 3h^2 + \frac{9}{4} h^3\right)\right)$$

$$= x[i] \underbrace{\left(1 + 3h + \frac{3^2 h^2}{2} + \frac{3^2 h^3}{2} + \frac{3^4}{2^3} h^4\right)}_{A_{RK}}$$

$$= A_{RK} \cdot x[i]$$

$$\therefore \quad x[1] = A_{RK}\, 5$$

$$x[2] = A_{RK} A_{RK} \cdot 5 = A_{RK}^2 5$$

$$\vdots$$

$$x[k] = A_{RK}^k \cdot 5$$

3. $x(t) = e^{3t} x(0) = 5 e^{3t}$

4. See plots, next page, from Andrew Lamkin.

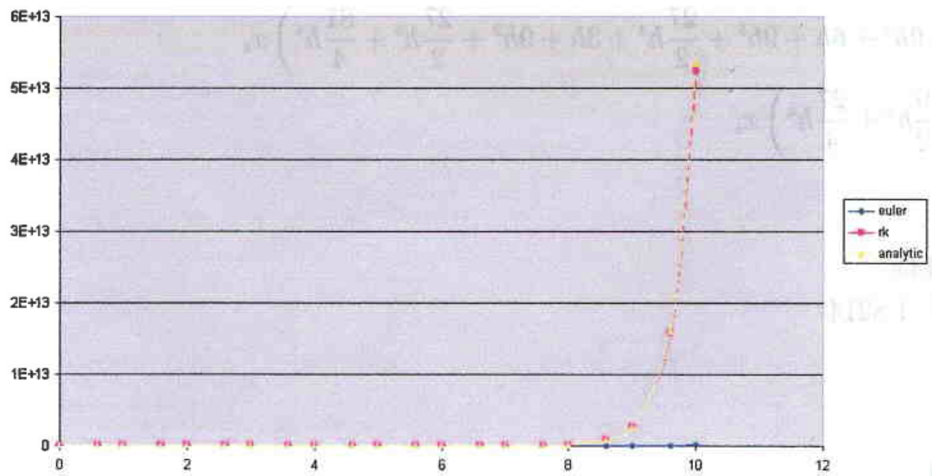



**Figure 2: Problem 4, Euler, RK4, and analytic solutions**

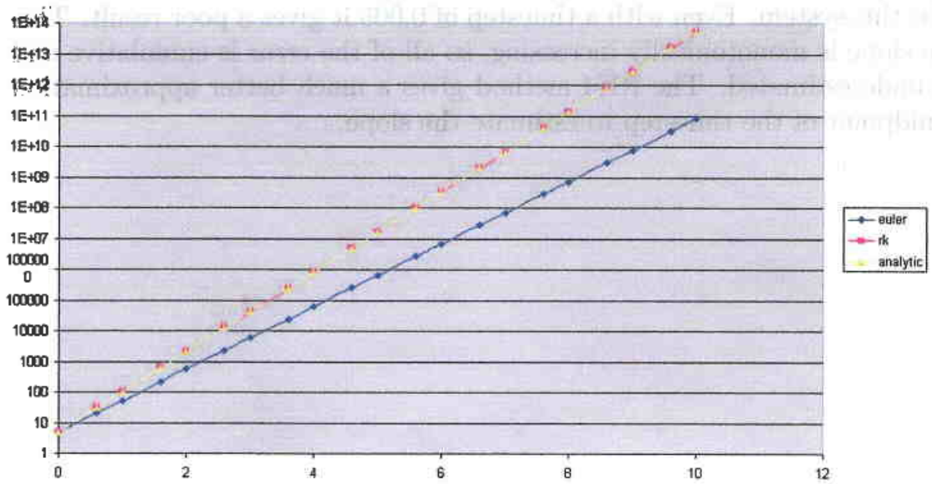



**Figure 3: Problem 4, Euler, RK4, and analytic solutions, log scale**