Problem Set #2 Solutions 10/15/09. EECE 568.

$\boxed{1.}$ $A = \begin{bmatrix} 0 & 1 \\ -2 & -2 \end{bmatrix} \Longrightarrow |\lambda I - A| = \begin{vmatrix} \lambda & -1 \\ 2 & \lambda+2 \end{vmatrix} = \lambda^2 + 2\lambda + 2 = 0$

$\lambda_{1,2} = -1 \pm j$

1.

$(A - \lambda_1 I)\, v_1 = 0$

$\begin{bmatrix} 1-j & 1 \\ -2 & -1-j \end{bmatrix} \begin{bmatrix} v_{11} \\ v_{12} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Longrightarrow v_1 = \begin{bmatrix} 1+j \\ -2 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 1-j \\ -2 \end{bmatrix}$

$V = [v_1 \ v_2] = \begin{bmatrix} 1+j & 1-j \\ -2 & -2 \end{bmatrix}. \quad V^{-1} = \begin{bmatrix} -j/2 & -1/4 - 1/4\, j \\ +j/2 & -1/4 + 1/4\, j \end{bmatrix}$

Modal form:

$\dot{z} = \bar{A} z + \bar{B} u,$

$y = \bar{C} z + \bar{D} u,$

$z = V^{-1} x$

$\bar{A} = \begin{bmatrix} -1+j & 0 \\ 0 & -1-j \end{bmatrix}$

$\bar{B} = V^{-1} B = \begin{bmatrix} -1/4 - 3/4\, j \\ -1/4 + 3/4\, j \end{bmatrix}$

$\bar{C} = CV = [-4 + 2j \quad -4 - 2j]$

$\bar{D} = D = 0.$

Real modal form:

$\dot{\tilde{x}} = \tilde{A} \tilde{x} + \tilde{B} u$

$y = \tilde{C} \tilde{x} + \tilde{D} u$

$\tilde{x} = Q^{-1} x$

$\tilde{A} = \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix}$ for $\lambda = \alpha \pm \beta j$, $\tilde{A} = \begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix}$

$\tilde{B} = Q^{-1} B = \begin{bmatrix} -1/2 \\ 3/2 \end{bmatrix}$

$\tilde{C} = CQ = [-4 \quad 2], \quad \tilde{D} = D = 0.$

$Q = [\mathrm{Re}(v_1)\ \mathrm{Im}(v_1)], \quad Q^{-1} = \begin{bmatrix} 0 & -1/2 \\ 1 & 1/2 \end{bmatrix}$

$= \begin{bmatrix} 1 & 1 \\ -2 & 0 \end{bmatrix}$

2. $H(s) = C(sI-A)^{-1}B + D$

$$= \begin{bmatrix} 2 & 3 \end{bmatrix} \begin{bmatrix} s & -1 \\ 2 & s+2 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + 0$$

$$= \frac{1}{s^2+2s+2} \begin{bmatrix} 2 & 3 \end{bmatrix} \begin{bmatrix} s+2 & 1 \\ -2 & s \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$= \frac{1}{s^2+2s+2} \begin{bmatrix} 2s-2 & 2+3s \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{5s}{s^2+2s+2}$$

3. A variety of sol'ns are possible: Laplace transforms, convolution, state-transition matrix via diagonalization, Jordan form, or infinite sum.

On the next pages, sol'ns from Erik Michling & Sara Farajian.

1.3) Compute the step response of the LTI system using two different methods. (assume $x(0) = [0,0]^T$).

① Frequency domain :

$$\hat{y}(s) = C(sI - A)^{-1} x(0) + \hat{H}(s)\,\hat{u}(s)$$
$$= \hat{H}(s)\,\hat{u}(s) \qquad (\text{since } x(0) = [0,0]^T)$$

$\hat{u}(s) = 1/s$ unit step

$$\Rightarrow \hat{y}(s) = \frac{5s}{s^2+2s+2} \cdot \frac{1}{s} = \frac{5}{s^2+2s+2} = \frac{5}{(s-(-1+i))(s-(-1-i))}$$

$$= \frac{A}{s+1-i} + \frac{B}{s+1+i}$$

$\Rightarrow As + A + Ai + Bs + B - Bi = 5$

$\Rightarrow A + B = 0$

$\Rightarrow (1+i)A + (1-i)B = 5$

So, $A = -2.5i$, $B = 2.5i$

$$\Rightarrow \hat{y}(s) = \frac{-2.5i}{s+1-i} + \frac{2.5i}{s+1+i}$$

$y(t) = \mathcal{L}^{-1}\{\hat{y}(s)\}$ since $\mathcal{L}^{-1}\left\{\frac{1}{s+a}\right\} = e^{-at}u(t)$

$$= -2.5i\left(e^{-(1-i)t}u(t)\right) + 2.5i\left(e^{-(1+i)t}u(t)\right)$$

$$= -2.5i\,e^{-t}e^{it}u(t) + 2.5i\,e^{-t}e^{-it}u(t)$$

$$= 2.5i \cdot e^{-t}u(t)\left[\underbrace{e^{-it} - e^{it}}_{-2i\sin(t)}\right]$$

$$\boxed{y(t) = 5e^{-t}\sin(t)\,u(t)}$$

② Time domain

$$h(t) = \mathcal{L}^{-1}\{\hat{H}(s)\} = \mathcal{L}^{-1}\left\{\frac{5s}{s^2+2s+2}\right\}$$

Aside: $\frac{5s}{s^2+2s+2} = \frac{5s}{(s-(-1+i))(s-(-1-i))} = \frac{A}{(s-(-1+i))} + \frac{B}{(s-(-1-i))}$

$$\Rightarrow As + A + Ai + Bs + B - Bi = 5s$$

$$\Rightarrow \begin{array}{l} A + B = 5 \\ (1+i)A + (1-i)B = 0 \end{array} \Rightarrow \begin{array}{l} A = \frac{5}{2}(1+i) \\ B = \frac{5}{2}(1-i) \end{array}$$

So,

$$h(t) = \mathcal{L}^{-1}\left\{\frac{5}{2}\left(\frac{1+i}{s+(1-i)}\right)\right\} + \mathcal{L}^{-1}\left\{\frac{5}{2}\left(\frac{1-i}{s+(1+i)}\right)\right\}$$

$$= \frac{5}{2}\left[(1+i)\,e^{-(1-i)t}u(t) + (1-i)\,e^{-(1+i)t}u(t)\right]$$

$$= \frac{5}{2}e^{-t}u(t)\Big[\underbrace{(1+i)e^{it} + (1-i)e^{-it}}_{e^{it}+e^{-it}+ie^{it}-ie^{-it} \,=\, 2\cos(t)-2\sin(t)}\Big]$$

$$= \frac{5}{2}e^{-t}u(t)\left[2\cos(t) - 2\sin(t)\right]$$

$$h(t) = 5e^{-t}u(t)\left[\cos(t) - \sin(t)\right]$$

Now,

$$y(t) = \int_{-\infty}^{\infty} g(\tau)h(t-\tau)\,d\tau = \int_{-\infty}^{\infty} h(\tau)g(t-\tau)\,d\tau$$

$$= 5\int_{-\infty}^{\infty} e^{-\tau}u(\tau)\left[\cos(\tau) - \sin(\tau)\right]u(t-\tau)\,d\tau$$

$$= 5\int_0^{\infty} e^{-\tau}\cos(\tau)\,u(t-\tau)\,d\tau - 5\int_0^{\infty} e^{-\tau}\sin(\tau)\,u(t-\tau)\,d\tau$$

$$= 5\int_0^{t} e^{-\tau}\cos(\tau)\,d\tau - 5\int_0^{t} e^{-\tau}\sin(\tau)\,d\tau$$

$$= 5\int_0^{t} e^{-\tau}\left(\frac{e^{i\tau}+e^{-i\tau}}{2}\right)d\tau - 5\int_0^{t} e^{-\tau}\left(\frac{e^{i\tau}-e^{-i\tau}}{2i}\right)d\tau$$

$$= \left(\frac{5}{2}-\frac{5}{2i}\right)\int_0^{t} e^{(-1+i)\tau}\,d\tau + \left(\frac{5}{2}+\frac{5}{2i}\right)\int_0^{t} e^{(-1-i)\tau}\,d\tau$$

$$= \left(\frac{5}{2}-\frac{5}{2i}\right)\cdot\frac{1}{-1+i}\underbrace{\left(e^{(-1+i)\tau}\Big|_0^t\right)}_{e^{-t}e^{it}-1} + \left(\frac{5}{2}+\frac{5}{2i}\right)\cdot\frac{1}{-1-i}\underbrace{\left(e^{(-1-i)\tau}\Big|_0^t\right)}_{e^{-t}e^{-it}-1}$$

$$= 2.5i\left(e^{-t}e^{it}-1\right) + 2.5i\left(e^{-t}e^{-it}-1\right)$$

$$= -2.5ie^{-t}e^{it} + 2.5i + 2.5ie^{-t}e^{-it} - 2.5i$$

$$= 2.5i\,e^{-t}\underbrace{\left(e^{-it}-e^{it}\right)}_{-2i\sin(t)} = \boxed{5e^{-t}\sin(t)}$$

method two :

solution for a LTI system : $\phi(t) = e^{A(t-t_0)} x_0 + \int_{t_0}^{t} e^{A(t-s)} B(s)u(s)ds$

we have $x(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

so $\rightarrow$ $\tilde{\phi}(t) = \int_0^t e^{\tilde{A}(t-s)} \tilde{B}\, ds$

and we have : $e^{\tilde{A}t} = e^{\begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix}} = e^{\alpha t} \begin{bmatrix} \cos\beta t & \sin\beta t \\ -\sin\beta t & \cos\beta t \end{bmatrix}$

and $\alpha = -1$, $\beta = 1$

so $\tilde{\phi}(t) = \int_0^t e^{-(t-s)} \begin{bmatrix} \cos(t-s) & \sin(t-s) \\ -\sin(t-s) & \cos(t-s) \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} ds$

$$= \int_0^t e^{-(t-s)} \begin{bmatrix} \cos(t-s) + 2\sin(t-s) \\ -\sin(t-s) + 2\cos(t-s) \end{bmatrix} ds$$

$$= \begin{bmatrix} \int_0^t e^{-(t-s)} \cos(t-s)\, ds + 2\int_0^t e^{-(t-s)} \sin(t-s)\, ds \\ \int_0^t e^{-(t-s)} (-\sin(t-s))\, ds + 2\int_0^t e^{-(t-s)} \cos(t-s)\, ds \end{bmatrix}$$

$t - s = \tau \rightarrow ds = -d\tau$ $\rightarrow$ $\int_0^t ds = -\int_t^0 d\tau = \int_0^t d\tau$

$s = 0 \rightarrow \tau = t$

$s = t \rightarrow \tau = 0$

so $\rightarrow$ $\tilde{\phi}(t) = \begin{bmatrix} \int_0^t e^{-\tau} \cos\tau\, d\tau + 2\int_0^t e^{-\tau} \sin\tau\, d\tau \\ 2\int_0^t e^{-\tau} \cos\tau\, d\tau - \int_0^t e^{-\tau} \sin\tau\, d\tau \end{bmatrix} \begin{matrix} \rightarrow \text{I} \\ \rightarrow \text{II} \end{matrix}$

Since : $\int e^{ax} \sin bx\, dx = \frac{1}{a^2+b^2} e^{ax} [a \sin(bx) - b\cos(bx)]$

$\int e^{ax} \cos bx\, dx = \frac{1}{a^2+b^2} e^{ax} [a\cos(bx) + b\sin(bx)]$

here : $a = -1$, $b = 1$

(I) $\frac{1}{2}e^{-\tau}\left[-\cos\tau+\sin\tau\right]\Big|_0^t + 2\times\frac{1}{2}e^{-\tau}\left[-\sin\tau-\cos\tau\right]\Big|_0^t$

$= \frac{1}{2}\left[-e^{-t}\cos t+e^{-t}\sin t+1\right] + \left[-e^{-t}\sin t - e^{-t}\cos t+1\right]$

$= -\frac{3}{2}e^{-t}\cos t - \frac{1}{2}e^{-t}\sin t + \frac{3}{2}$

(II) $\cancel{2}\times\frac{1}{\cancel{2}}e^{-\tau}\left[-\cos\tau+\sin\tau\right]\Big|_0^t - \frac{1}{2}e^{-\tau}\left[-\sin\tau-\cos\tau\right]\Big|_0^t$

$= \left[-e^{-t}\cos t+e^{-t}\sin t+1\right] - \frac{1}{2}\left[-e^{-t}\sin t - e^{-t}\cos t+1\right]$

$= -\frac{1}{2}e^{-t}\cos t + \frac{3}{2}e^{-t}\sin t + \frac{1}{2}$

$\Longrightarrow \quad \tilde{\Phi}(t) = \tilde{x}(t) = \begin{bmatrix} -\frac{3}{2}e^{-t}\cos t - \frac{1}{2}e^{-t}\sin t + \frac{3}{2} \\ -\frac{1}{2}e^{-t}\cos t + \frac{3}{2}e^{-t}\sin t + \frac{1}{2} \end{bmatrix}$

$x(t) = V\tilde{x}(t) = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} -\frac{3}{2}e^{-t}\cos t - \frac{1}{2}e^{-t}\sin t + \frac{3}{2} \\ -\frac{1}{2}e^{-t}\cos t + \frac{3}{2}e^{-t}\sin t + \frac{1}{2} \end{bmatrix}$

$= \begin{bmatrix} -\frac{3}{2}e^{-t}\cos t - \frac{1}{2}e^{-t}\sin t + \frac{3}{2} \\ e^{-t}\cos t + 2e^{-t}\sin t - 1 \end{bmatrix}$

$\rightarrow y(t) = C\,x(t) = \begin{bmatrix} 2 & 3 \end{bmatrix}\begin{bmatrix} -\frac{3}{2}e^{-t}\cos t - \frac{1}{2}e^{-t}\sin t + \frac{3}{2} \\ e^{-t}\cos t + 2e^{-t}\sin t - 1 \end{bmatrix}$

$= -3e^{-t}\cos t - e^{-t}\sin t + 3 + 3e^{-t}\cos t + 6e^{-t}\sin t - 3$

$\rightarrow \quad \boxed{y(t) = 5e^{-t}\sin t}$ ✓

the same answer as in method one.

$\boxed{12}$ $$x[n] - A^n x[0] = [b \ Ab \ \cdots \ A^{n-1} b] \begin{bmatrix} u[n-1] \\ \vdots \\ u[0] \end{bmatrix}$$

The left-hand side of the equation can take any arbitrary value in $\mathbb{R}^n$. The left-hand side of the equation is the same as the range space of $[b \ Ab \ \cdots \ A^{n-1}b]$. For $R([b \cdots A^{n-1}b]) = \mathbb{R}^n$, the columns of $[b \cdots A^{n-1}b]$ must form a <u>basis</u> of $\mathbb{R}^n$ and hence must be linearly independent. An easy way to evaluate this is to check the <u>rank</u> of $[b \cdots A^{n-1}b]$, which is the <u>controllability matrix</u>.

$\boxed{13}$ $$\dot{x}(t) = A(t)x(t), \quad A(t) = \begin{bmatrix} A_{11}(t) & A_{12}(t) \\ 0 & A_{22}(t) \end{bmatrix}$$

The state transition matrix satisfies $x(t) = \Phi(t, t_0)x(t_0)$.

So $$\dot{x}(t) = \frac{d}{dt}\Phi(t, t_0)x(t_0)$$
$$= \begin{bmatrix} \frac{d}{dt}\Phi_{11}(t,t_0) & \frac{d}{dt}\Phi_{12}(t,t_0) \\ \frac{d}{dt}\Phi_{21}(t,t_0) & \frac{d}{dt}\Phi_{22}(t,t_0) \end{bmatrix} x(t_0)$$

and Φ satisfies the diff. eqn $\dot{\Phi} = A\Phi$ for all $t \geq t_0$.

$$A(t)\Phi(t,t_0) = \begin{bmatrix} A_{11}(t)\Phi_{11}(t,t_0) + A_{12}(t)\Phi_{21}(t,t_0) & A_{11}(t)\Phi_{12}(t,t_0) + A_{22}(t)\Phi_{22}(t,t_0) \\ A_{22}(t)\Phi_{21}(t,t_0) & A_{22}(t)\Phi_{22}(t,t_0) \end{bmatrix}$$

First show $\Phi_{21}(t, t_0) = 0$

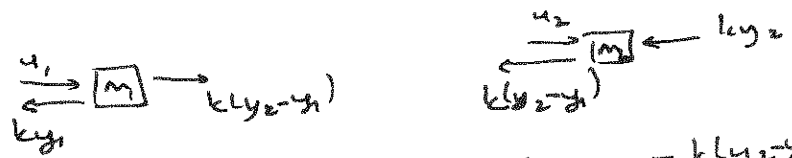
By method of successive approximations
$$\Phi_m(t,t_0) = I + \int_{t_0}^{t} A(s_m)ds_m + \cdots + \int_{t_0}^{t} A(s_m) \cdots \int_{t_0}^{s_2} A(s_1)ds_1 \cdots ds_m$$
operations on A will yield upper triangular matrices, hence $\Phi_m(t,t_0)$ is upper triangular. Since $\Phi(t,t_0) = \lim_{m\to\infty} \Phi_m(t,t_0)$

$\Phi(t, t_0)$ is also upper triangular, so $\Phi_{21}(t, t_0) = 0$.

Then $\dot{\Phi}(t, t_0) = \begin{bmatrix} A_{11}(t)\Phi_{11}(t, t_0) & A_{11}(t)\Phi_{12}(t, t_0) + A_{22}(t)\Phi_{22}(t, t_0) \\ 0 & A_{22}(t)\Phi_{22}(t, t_0) \end{bmatrix}$.

"+" is $\rightarrow$.

$\boxed{4}$.



$m_2\ddot{y}_2 = u_2 - k(y_2-y_1) - ky_2$

$m_1\ddot{y}_1 = u_1 - ky_1 + k(y_2-y_1)$

1. $\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\begin{bmatrix} \ddot{y}_1 \\ \ddot{y}_2 \end{bmatrix} = \begin{bmatrix} -2k & k \\ k & -2k \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$

With $x = \begin{bmatrix} y_1 \\ y_2 \\ \dot{y}_1 \\ \dot{y}_2 \end{bmatrix}$, $u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$, $y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$

$$\begin{bmatrix} \ddot{y}_1 \\ \ddot{y}_2 \end{bmatrix} = M^{-1}K\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + M^{-1}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$\therefore \quad \dot{x} = \begin{bmatrix} 0 & I \\ -M^{-1}K & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ M^{-1} \end{bmatrix} u$$

$$y = \begin{bmatrix} I & 0 \end{bmatrix} x + [0]\, u.$$

2. See attached plot, next page.

a) $\tilde{x} = Q^{-1}x_0$ where Q is matrix of eigenvectors

Since eigenvalues are imaginary: $\lambda = \pm j, \pm\sqrt{3}j$

choose real modal form to interpret representation of x_0 as transformation to modal coordinates.

Therefore $AV = V\Lambda$, $\Lambda = \begin{bmatrix} 0 & 1 & & \\ -1 & 0 & & \\ & & 0 & \sqrt{3} \\ & & -\sqrt{3} & 0 \end{bmatrix}$, $V = \begin{bmatrix} | & | & | & | \\ Re(v_1) & Im(v_1) & Re(v_3) & Im(v_3) \\ | & | & | & | \end{bmatrix}$

$$V_{1,2} = \begin{bmatrix} \pm j \\ \pm j \\ 1 \\ 1 \end{bmatrix} \qquad V_{3,4} = \begin{bmatrix} \pm 1/\sqrt{3}\, j \\ \mp 1/\sqrt{3}\, j \\ -1 \\ 1 \end{bmatrix}$$

$$\therefore\ V = \begin{bmatrix} 0 & 1 & 0 & 1/\sqrt{3} \\ 0 & 1 & 0 & -1/\sqrt{3} \\ 1 & 0 & -1 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix}, \quad \text{and} \quad V = Q$$

$$\therefore\ \tilde{x}_0 = V^{-1} x_0 = \begin{bmatrix} 0 \\ 1/2 \\ 0 \\ \sqrt{3}/2 \end{bmatrix}$$

b) There are 2 frequencies inherent to this system. The representation of x_0 in modal coordinates shows that there are non-zero initial conditions in 2 of the 4 modal coordinates. Hence we expect to see both natural frequencies actuated in the output.

3. First notice that $Q^{-1} \cdot \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$, hence only 1 mode would be actuated. Second, notice that x_0 is proportional to one of the eigenvectors. Since

$\dot{x} = Ax$, if x is proportional to an eigenvector, we would expect $\dot{x}$ to simply scale w/ x by a factor of λ.

we would expect $\dot{x}$ to simply scale w/ x by a factor of λ, at 1 frequency — a "mode".

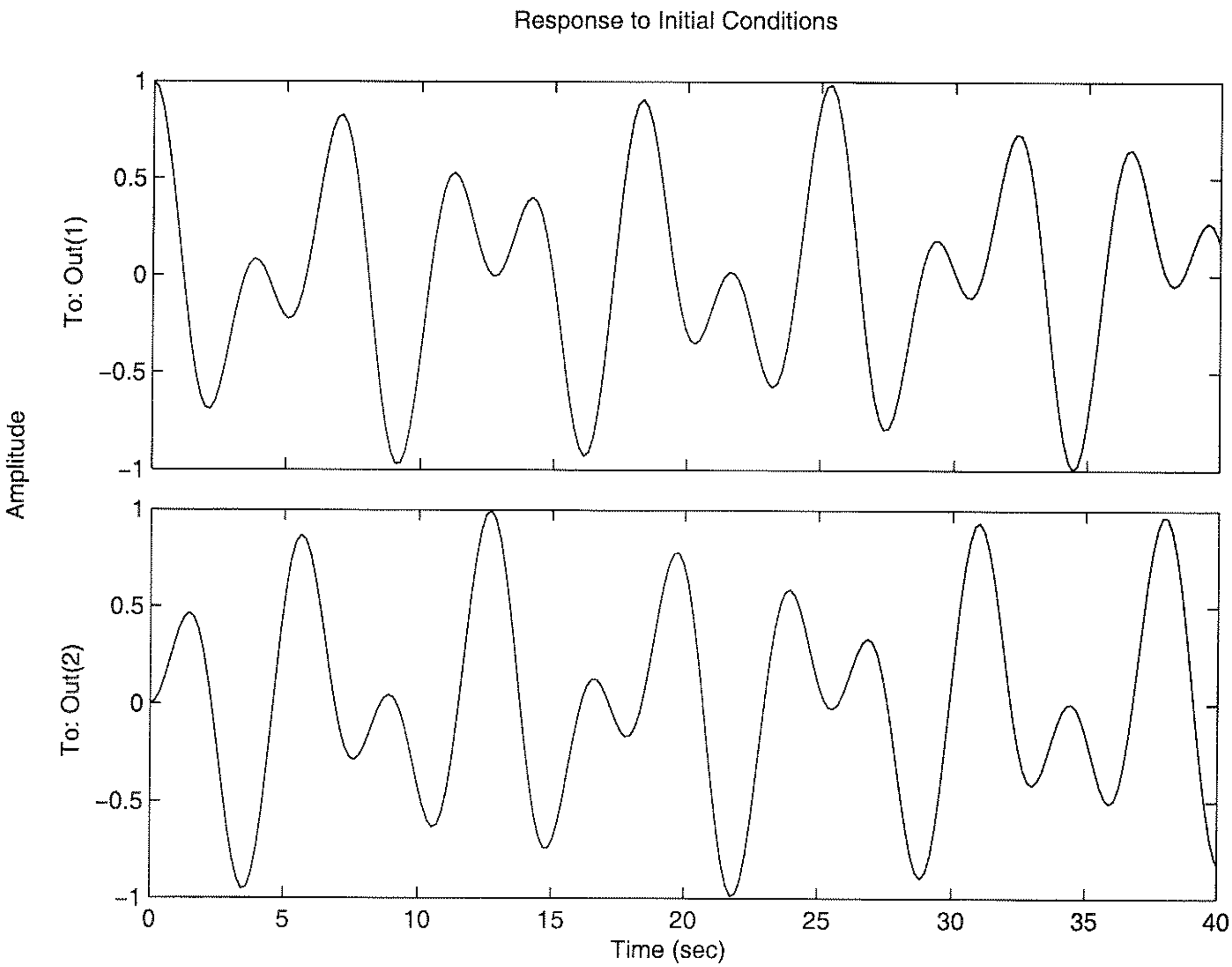
Hence we see simple oscillations

$$x(t) = e^{At} x_0 = V e^{\Lambda t} \underbrace{V^{-1} x_0}$$

yields a column of identity matrix for V chosen orthonormal.

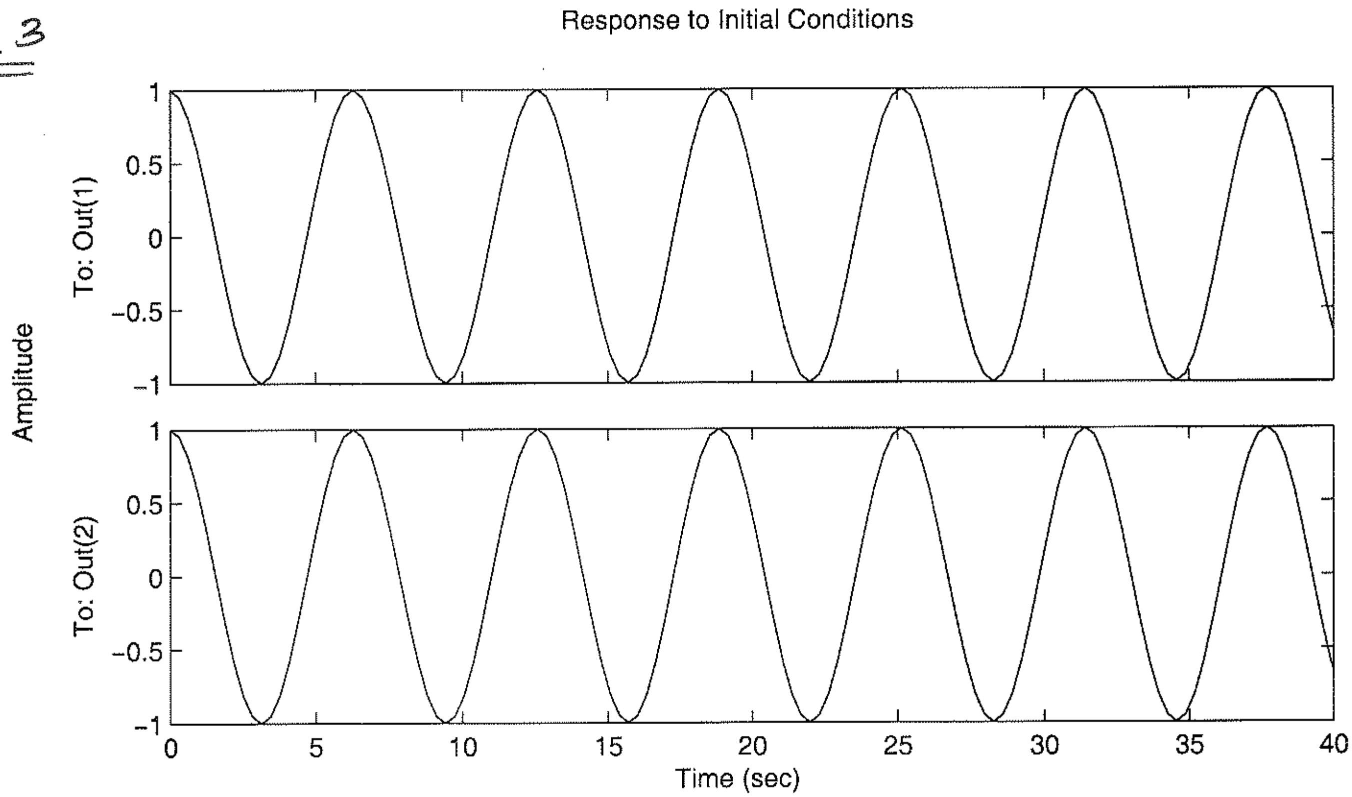
4. Another mode occurs when the masses start an equal distance apart in opposite directions e.g., $\begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}$ (as in 4th column of V, above).

4.2

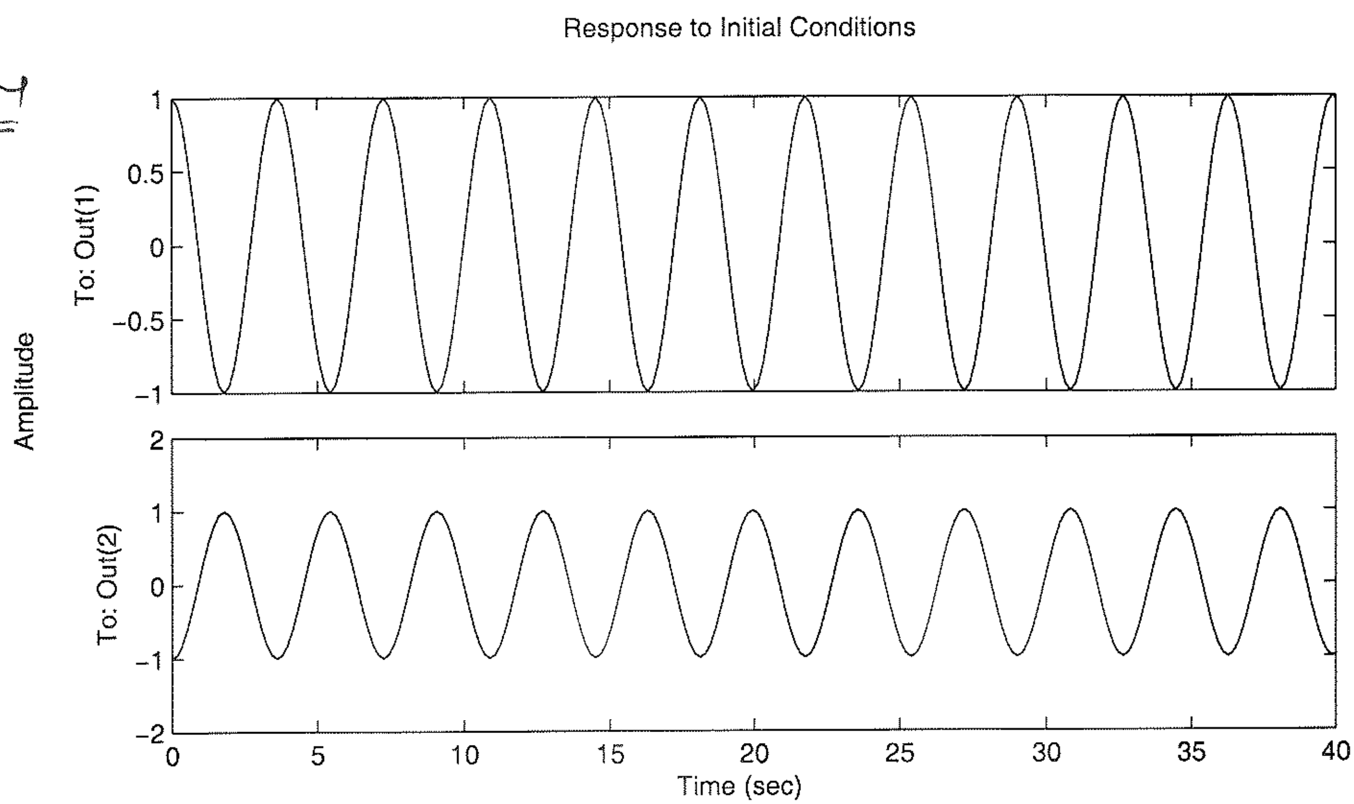


Response to Initial Conditions
Amplitude
To: Out(1)
To: Out(2)
1
0.5
0
−0.5
−1
0
5
10
15
20
25
30
35
40
Time (sec)

4.3


Response to Initial Conditions
Amplitude
To: Out(1)
To: Out(2)
Time (sec)
1
0.5
0
−0.5
−1
0
5
10
15
20
25
30
35
40

4.4


Response to Initial Conditions
Amplitude
To: Out(1)
To: Out(2)
Time (sec)
1
0.5
0
−0.5
−1
2
−2
0
5
10
15
20
25
30
35
40