$\boxed{1}$ eigenvalue of A: $-0.9, -1, -1$

only 2 independent eigenvectors, $\therefore$ A is defective.

1. $\dot{x} = Ax$ is asymptotically stable since $Re(\lambda_i) < 0$ for $i \in \{1, 2, 3\}$.

2. $x[k+1] = Ax[k]$ is unstable since although $|\lambda_i| \le 1$, the repeated eigenvalue at $\lambda = 1$ (on the unit disk) have only 1 independent eigenvector.

3. $\dot{x} = Ax + Bu$, $y = Cx$ is BIBO stable since $\dot{x} = Ax$ is exponentially stable.

4. More information about B & C is required to deduce BIBO stability.

$\boxed{2}$

1. $\int_0^\infty |H_{11}(t)|\,dt = \int_0^\infty te^{-t}\,dt$

$\int u\,dv = uv - \int v\,du$

$u = t$, $e^{-t}dt = dv$

$du = dt$, $-e^{-t} = v$

$= -te^{-t}\Big|_0^\infty - \int_0^\infty (-e^{-t})\,dt$

$= 0 + -e^{-t}\Big|_0^\infty$

$= +1 < \infty \Rightarrow$ BIBO stable

but $\int_0^\infty \left|\frac{1}{1+t}\right| dt = \ln(1+t)\Big|_0^\infty = \infty \rightarrow \therefore$ not BIBO stable.

For $H(t)$ to be BIBO stable, $H_{11}(t)$ & $H_{12}(t)$ must be BIBO stable.

$\therefore$ BIBO <u>unstable</u>.

2. $u(t) = C\frac{dy}{dt} + i_L$, $y(t) = L\frac{di_L}{dt}$

$U(s) = CsY(s) + I(s)$ $Y(s) = LsI(s)$

$U(s) = Y(s)\left(Cs + \frac{1}{Ls}\right)$ $\frac{Y(s)}{Ls} = I(s)$

$\frac{Y(s)}{U(s)} = \frac{Ls}{LCs^2+1} = \frac{1}{C}\frac{s}{s^2+\frac{1}{LC}}$

for $L = C = 1$,

$$H(s) = \frac{s}{s^2+1}$$

For $u(t) = \sin t$

$$Y(s) = H(s)U(s) = \frac{s}{s^2+1}\cdot\frac{1}{s^2+1} = \frac{s}{(s^2+1)^2}$$

$$\Rightarrow y(t) = \frac{1}{2}t\sin t$$

$\therefore$ a bounded input produce an unbounded output.

$\boxed{3}$ $\dot{x} = \begin{bmatrix} -1 & 0 \\ -e^{3t} & 0 \end{bmatrix} x$

1. For $t_0 = 0$,

$$\dot{x}_1 = -x_1 \implies x_1(t) = e^{-t} x_1(0)$$

$$\dot{x}_2 = -e^{-3t} x_1$$

$$= -e^{-3t} e^{-t} x_1(0)$$

$$\int_{t_0}^{t} \dot{x}_2 dt = \int_{t_0}^{t} -e^{-4t} x_1(0) dt = \frac{1}{4} e^{-4t} x_1(0) \Big|_0^t$$

$$\implies x_2(t) = \frac{1}{4}(e^{-4t} - 1) x_1(0) + x_2(0)$$

Choose $t_0 = 0$, $x(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \implies x(t) = \begin{bmatrix} e^{-t} \\ \frac{1}{4}(e^{-4t} - 1) \end{bmatrix}$

$x(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \implies x(t) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Fundamental matrix $X(t) = \begin{bmatrix} e^{-t} & 0 \\ \frac{1}{4}(e^{-4t} - 1) & 1 \end{bmatrix}$

then $X^{-1}(t) = \frac{1}{e^{-t} - 0} \begin{bmatrix} 1 & 0 \\ -\frac{1}{4}(e^{-4t} - 1) & e^{-t} \end{bmatrix} = \begin{bmatrix} e^{t} & 0 \\ -\frac{e^{t}}{4}(e^{-4t} - 1) & 1 \end{bmatrix}$

$$\therefore \Phi(t, t_0) = X(t) X^{-1}(t_0)$$

$$= \begin{bmatrix} e^{-t} & 0 \\ \frac{1}{4}(e^{-4t} - 1) & 1 \end{bmatrix} \begin{bmatrix} e^{t_0} & 0 \\ -\frac{1}{4}(e^{-4t_0} - 1) e^{t_0} & 1 \end{bmatrix}$$

$$= \begin{bmatrix} e^{-(t - t_0)} & 0 \\ \frac{e^{t_0}}{4}(e^{-4t} - e^{-4t_0}) & 1 \end{bmatrix}$$

2. Since $\|\Phi(t, t_0)\|$ is bounded (all elements bounded by finite #s) system is stable, but not asymptotically stable (not all elements converge to 0)

Question 4.

part 1. $A = \begin{bmatrix} 0 & 1 \\ -\frac{1}{2} & -1 \end{bmatrix}$, $P = \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix}$, $Q = \begin{bmatrix} q_1 & q_2 \\ q_2 & q_3 \end{bmatrix}$

$$A^T P + PA = \begin{bmatrix} 0 & -\frac{1}{2} \\ 1 & -1 \end{bmatrix}\begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} + \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -\frac{1}{2} & -1 \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{1}{2}p_2 & -\frac{1}{2}p_3 \\ p_1 - p_2 & p_2 - p_3 \end{bmatrix} + \begin{bmatrix} -\frac{1}{2}p_2 & p_1 - p_2 \\ -\frac{1}{2}p_3 & p_2 - p_3 \end{bmatrix}$$

$$= \begin{bmatrix} -p_2 & -\frac{1}{2}p_3 + p_1 - p_2 \\ -\frac{1}{2}p_3 + p_1 - p_2 & 2p_2 - 2p_3 \end{bmatrix} = \begin{bmatrix} q_1 & q_2 \\ q_2 & q_3 \end{bmatrix}$$

$$q_1 = -p_2 \rightarrow \boxed{p_2 = -q_1}$$

$$2p_2 - 2p_3 = -2q_1 - 2p_3 = q_3 \rightarrow \boxed{p_3 = -q_1 - \frac{q_3}{2}}$$

$$-\frac{1}{2}p_3 + p_1 - p_2 = -\frac{1}{2}\left[-q_1 - \frac{1}{2}q_3\right] + p_1 + q_1 = q_2 \rightarrow \boxed{p_1 = q_2 - \frac{3}{2}q_1 - \frac{1}{4}q_3}$$

* Q is negative definite $\rightarrow \begin{cases} q_1 q_3 - q_2^2 > 0 & \rightarrow |q_2| < \sqrt{q_1 q_3} \\ q_1, q_3 < 0 \end{cases}$ ok.

* P is positive definite $\rightarrow \begin{cases} p_1 p_3 - p_2^2 > 0 \\ p_1 > 0 \end{cases}$

(I) $p_2 = -q_1 \xrightarrow{q_1 < 0} p_2 > 0$

(II) $p_3 = -q_1 - \frac{q_3}{2} \xrightarrow{q_1, q_3 < 0} p_3 > 0$

(II) $\Delta = P_1P_3 - P_2^2 = \left[q_2 - \frac{3}{2}q_1 - \frac{1}{4}q_3\right]\left[-q_1 - \frac{1}{2}q_3\right] - \left[-q_1^2\right]$

$= -q_1q_2 - \frac{1}{2}q_2q_3 + \frac{3}{2}q_1^2 + \frac{3}{4}q_1q_3 + \frac{1}{4}q_1q_3 + \frac{1}{8}q_3^2 - q_1^2$

$= -q_1q_2 - \frac{1}{2}q_2q_3 + \frac{1}{2}q_1^2 + q_1q_3 + \frac{1}{8}q_3^2$

$\rightarrow \Delta = q_2\underbrace{\left[-q_1 - \frac{1}{2}q_3\right]}_{>0} + \underbrace{\frac{1}{2}q_1^2}_{>0} + \underbrace{q_1q_3}_{>0} + \underbrace{\frac{1}{8}q_3^2}_{>0}$

if $q_2 > 0 \rightarrow \Delta > 0$ ✓

if $q_2 < 0$, since $|q_2| < \sqrt{q_1q_3}$ worst case is $q_2 = -\sqrt{q_1q_3}$ ✓

so $\Delta = \sqrt{q_1q_3}\left[q_1 + \frac{1}{2}q_3\right] + \frac{1}{2}\left[q_1 + \frac{1}{2}q_3\right]^2 + \frac{1}{2}q_1q_3 > 0$

$\rightarrow \left[q_1 + \frac{1}{2}q_3\right]^2 + q_1q_3 + 2\sqrt{q_1q_3}\left[q_1 + \frac{1}{2}q_3\right] > 0$

$\rightarrow \left[\left(q_1 + \frac{1}{2}q_3\right) + \sqrt{q_1q_3}\right]^2 > 0$ ✓ ✓

8pd 10/16.