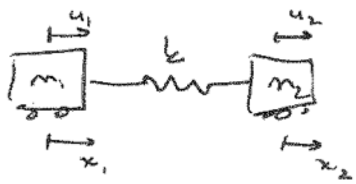
Problem Set #4 EECE 568 M. Qoshi

$\boxed{1}$



$m_1 = m_2 = k = 1.$

$$\dot{x} = Ax + Bu, \qquad A = \left[\begin{array}{c|c} 0 & I \\ \hline -M^{-1}K & 0 \end{array}\right] = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \end{bmatrix}$$
$$y = Cx$$

1. $B_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \Rightarrow \mathcal{C} = [B_1 \ AB_1 \ A^2B_1 \ A^3B_1] = \begin{bmatrix} 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$

rank$(\mathcal{C}) = 4 \Rightarrow$ controllable w/ motor on 1st car only.

$B_2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \Rightarrow \mathcal{C} = [B_2 \ AB_2 \ A^2B_2 \ A^3B_2] = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & -1 \\ 1 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & -1 & 0 & 0 \end{bmatrix}$

rank$(\mathcal{C}) = 4 \Rightarrow$ controllable w/ both motors.

2. $C_1 = [1 \ 0 \ 0 \ 0] \Rightarrow \mathcal{O} = \begin{bmatrix} C_1 \\ AC_1 \\ A^2C_1 \\ A^3C_1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}$

rank$(\mathcal{O}) = 4 \Rightarrow$ observable by measuring posn of 1st car only.

$C_2 = [0 \ 1 \ 0 \ 0] \Rightarrow \mathcal{O} = \begin{bmatrix} C_2 \\ AC_2 \\ A^2C_2 \\ A^3C_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}$

rank$(\mathcal{O}) = 4 \Rightarrow$ observable by measuring velocity of 1st car only.

3*. $K = \text{place}(A, B_1, P)$ where $P = \begin{bmatrix} -1-i \\ -1+i \\ -100+100i \\ -100-100i \end{bmatrix}$

is $K = [20400 \quad 19600 \quad 202 \quad 40198]$

4*. $u = -Kx \Rightarrow \dot{x} = (A - BK)x$ for closed-loop system.

See attached plots.

$\boxed{2}$ $\mathcal{C} = [B \ AB \ A^2B] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$, $\text{rank}(\mathcal{C}) = 1 = n_c$

1.

$R(\mathcal{C}) = \text{span}\left\{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right\}$ (range of controllability matrix is controllable subspace)

$\mathcal{O} = \begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & \omega & 0 \\ 0 & 0 & 0 \\ 0 & \omega^2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $\text{rank}(\mathcal{O}) = 3 = n_o$

$\Longrightarrow$ observable.

Let $Q = \begin{bmatrix} \underbrace{\begin{matrix} 0 \\ 0 \\ 1 \end{matrix}}_{\text{spans controllable subspace}} & \underbrace{\begin{matrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{matrix}}_{\text{spans uncontrollable subspace.}} \end{bmatrix}$, then $Q^{-1} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

Then $\hat{A} = Q^{-1}AQ$, $\hat{B} = Q^{-1}B$, $\hat{C} = CQ$, $\hat{D} = D$

$= \left[\begin{array}{c|cc} 0 & 0 & 0 \\ \hline 0 & 0 & \omega \\ 0 & -\omega & 0 \end{array}\right]$ $= \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ $= \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ $= [0]$

2. $\text{eig}(A_c) = 0$, $\text{eig}(A_{uc}) = \pm j\omega$. Since eigenvalues of A_{uc} cannot be changed, it is NOT possible for closed-loop system to have eigvals at $-1 \pm j, -2$.

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3. $\begin{cases} \dot{x}_c = A_c x_c + B_c u_c \\ y_c = C_c x_c + D_c u \end{cases}$ will have same t.f. as $\dot{x} = Ax + Bu$, $y = Cx + Du$.

$$= \begin{cases} \dot{x}_c = 0 \cdot x_c + 1 \cdot u_c \\ y = 1 \cdot x_c + 0 \cdot u \end{cases}$$

$$\hat{G}(s) = C_c (sI - A_c)^{-1} B_c + D_c$$

$$= 1 (s - 0)^{-1} \cdot 1 + 0$$

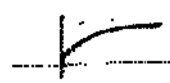
$$= \frac{1}{s}$$

$$G(s) = C (sI - A)^{-1} B + D$$

$$= \hat{G}(s).$$

Note: $\hat{G}(s)$ is actually 2×1 since there are 2 outputs, but other element is 0.

4*. With controllable pole moved from $\lambda = 0$ to $\lambda = -2$, the closed-loop system will have eigenvalues at $\lambda = -2, \pm j\omega$ & will be stable in the sense of Lyapunov. Because the dynamics are decoupled, the oscillatory component will not decay.

The zero-state response to a step input will show a 1st-order system response (e.g., no oscillations)  in the controllable state. The uncontrollable states will have no dynamics — remain at 0.

$\boxed{4.}$

1. The system is unstable, its eigenvalues $-.75 \pm 2.22j$, 0
2. The controllability matrix is rank 1, hence the complex pole pair is uncontrollable & the single pole at 0 is controllable. So the system can never be asymptotically stable in closed-loop operation — at best, just stable in the sense of Lyapunov.

3 System design

Solution from Andrew Larkin

3.1

We note that

$$\mathbf{AB} = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \\ b_{41} & b_{42} \\ b_{51} & b_{52} \end{bmatrix} = \begin{bmatrix} b_{11}+b_{21} & b_{12}+b_{22} \\ b_{21} & b_{22} \\ b_{31}+b_{41} & b_{32}+b_{42} \\ b_{41} & b_{42} \\ b_{51} & b_{52} \end{bmatrix} = \mathbf{B} + \tilde{\mathbf{B}} \qquad \text{where } \tilde{\mathbf{B}} = \begin{bmatrix} b_{21} & b_{22} \\ 0 & 0 \\ b_{41} & b_{42} \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

In general, $\mathbf{A}^k\mathbf{B} = \mathbf{B} + k\tilde{\mathbf{B}}$ and $\mathcal{C} = \begin{bmatrix} \mathbf{B} & \mathbf{AB} & \mathbf{A}^2\mathbf{B} & \mathbf{A}^3\mathbf{B} & \mathbf{A}^4\mathbf{B} \end{bmatrix} = \begin{bmatrix} \mathbf{B} & \mathbf{B}+\tilde{\mathbf{B}} & \mathbf{B}+2\tilde{\mathbf{B}} & \mathbf{B}+3\tilde{\mathbf{B}} & \mathbf{B}+4\tilde{\mathbf{B}} \end{bmatrix}$. Then $\text{rank}(\mathcal{C}) = \text{rank}\left(\begin{bmatrix} \mathbf{B} & \mathbf{B}+\tilde{\mathbf{B}} & \mathbf{B}+2\tilde{\mathbf{B}} & \mathbf{B}+3\tilde{\mathbf{B}} & \mathbf{B}+4\tilde{\mathbf{B}} \end{bmatrix}\right) = \text{rank}\left(\begin{bmatrix} \mathbf{B} & \tilde{\mathbf{B}} & 0 & 0 & 0 \end{bmatrix}\right) \leq 4$, so the system cannot be controllable.

3.2

Similarly, $\mathbf{CA}^k = \mathbf{C} + k\tilde{\mathbf{C}}$ where $\tilde{\mathbf{C}} = \begin{bmatrix} 0 & c_{11} & 0 & c_{13} & 0 \\ 0 & c_{21} & 0 & c_{23} & 0 \\ 0 & c_{31} & 0 & c_{33} & 0 \end{bmatrix}$, and

$$\text{rank}(\mathcal{O}) = \text{rank}\left(\begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \\ \mathbf{CA}^2 \\ \mathbf{CA}^3 \\ \mathbf{CA}^4 \end{bmatrix}\right) = \text{rank}\left(\begin{bmatrix} \mathbf{C} \\ \mathbf{C}+\tilde{\mathbf{C}} \\ \mathbf{C}+2\tilde{\mathbf{C}} \\ \mathbf{C}+3\tilde{\mathbf{C}} \\ \mathbf{C}+4\tilde{\mathbf{C}} \end{bmatrix}\right) = \text{rank}\left(\begin{bmatrix} \mathbf{C} \\ \tilde{\mathbf{C}} \\ 0 \\ 0 \\ 0 \end{bmatrix}\right) = r_c$$

where r_c is the total number of independent rows of $\mathbf{C}$ and $\tilde{\mathbf{C}}$. If $r_c \geq 5$, then the system is observable.

Solution from Andrew Larkin

5 Decoupled system

5.1

$$\mathbf{A} = \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{AB} = \begin{bmatrix} 1 \\ \lambda \\ 0 \end{bmatrix}, \quad \mathbf{A}^2\mathbf{B} = \begin{bmatrix} 2\lambda \\ \lambda^2 \\ 0 \end{bmatrix}, \quad \mathcal{C} = \begin{bmatrix} 0 & 1 & 2\lambda \\ 1 & \lambda & \lambda^2 \\ 0 & 0 & 0 \end{bmatrix}$$

Assuming $\lambda \neq 0$,

$$\text{range}(\mathcal{C}) = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$

$$\mathbf{C} = \begin{bmatrix} 0 & 1 & 1 \end{bmatrix}, \quad \mathbf{CA} = \begin{bmatrix} 0 & \lambda & 1+\lambda \end{bmatrix}, \quad \mathbf{CA}^2 = \begin{bmatrix} 0 & \lambda^2 & 2\lambda+\lambda^2 \end{bmatrix}, \quad \mathcal{O} = \begin{bmatrix} 0 & 1 & 1 \\ 0 & \lambda & 1+\lambda \\ 0 & \lambda^2 & 2\lambda+\lambda^2 \end{bmatrix}$$

$$\text{null}(\mathcal{O}) = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$$

Since $\text{rank}(\mathcal{C}) = 2 < 3$, the system is uncontrollable. Since $\text{rank}(\mathcal{O}) = 2 < 3$, the system is unobservable.

5.2

$$\mathbf{Q}_{C\bar{O}} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{Q}_{CO} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{Q}_{\bar{C}\bar{O}} = \phi, \quad \mathbf{Q}_{\bar{C}O} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{Q} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{Q}^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{Q}^{-1}\mathbf{AQ} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & \lambda & 1 \\ \lambda & 1 & 0 \\ 0 & 0 & \lambda \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \lambda & 0 & 1 \\ 1 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} = \begin{bmatrix} \mathbf{A}_{CO} & 0 & \mathbf{A}_{13} \\ \mathbf{A}_{21} & \mathbf{A}_{C\bar{O}} & 0 \\ 0 & 0 & \mathbf{A}_{\bar{C}O} \end{bmatrix}$$

$$\mathbf{Q}^{-1}\mathbf{B} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \mathbf{B}_1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{CQ} = \begin{bmatrix} 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \mathbf{C}_1 & 0 & 1 \end{bmatrix}$$

$$\dot{\tilde{\mathbf{x}}} = \begin{bmatrix} \lambda & 0 & 1 \\ 1 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} \tilde{\mathbf{x}} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u, \qquad y = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix} \tilde{\mathbf{x}}$$

5.3

The same transfer function will be obtained with just the controllable and observable part of the system:

$$\dot{\tilde{x}}_1 = \lambda \tilde{x}_1 + u \qquad y = \tilde{x}_1$$