Midterm Solutions EECE 568

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1. $\dot{x} = Ax + B_1 u + B_2 w_1 \qquad w_1 = H_1 z$

$= Ax + B_2 H_1 z + B_1 u$

$\dot{z} = Fz + G_1 v + G_2 w_2, \qquad w_2 = Cx + D_1 u + D_2 w_1$

$= Cx + D_1 u + D_2 H_1 z$

$= (F + G_2 D_2 H_1) z + G_2 C x + G_1 v + G_2 D_1 u$

$y = H_2 z + J w_2 = H_2 z + J D_2 H_1 z + JCx + JD_1 u$

$= (H_2 + J D_2 H_1) z + JCx + JD_1 u$

$$\begin{bmatrix} \dot{x} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} A & B_2 H_1 \\ G_2 C & F + G_2 D_2 H_1 \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix} + \begin{bmatrix} B_1 & 0 \\ G_2 D_1 & G_1 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \bar{A} \begin{bmatrix} x \\ z \end{bmatrix} + \bar{B} \begin{bmatrix} u \\ v \end{bmatrix}$$

$$y = \begin{bmatrix} JC & (H_2 + J D_2 H_1) \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix} + \begin{bmatrix} JD_1 & 0 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \bar{C} \begin{bmatrix} x \\ z \end{bmatrix} + \bar{D} \begin{bmatrix} u \\ v \end{bmatrix}$$

2. $h(t) = \bar{C} \int_0^t e^{\bar{A}(t-\tau)} \bar{B} \begin{bmatrix} 0 \\ \delta(t) \end{bmatrix} d\tau + \bar{D}\delta(t) = \bar{C} e^{\bar{A}t} \bar{B} + \bar{D} \begin{bmatrix} 0 \\ \delta(t) \end{bmatrix}$

$$= \begin{bmatrix} JC & H_2 + J D_2 H_1 \end{bmatrix} \begin{bmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{21} & \Phi_{22} \end{bmatrix} \begin{bmatrix} B_1 & 0 \\ G_2 D_1 & G_1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} JD_1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ \delta(t) \end{bmatrix}$$

$$= \begin{bmatrix} JC & H_2 + J D_2 H_1 \end{bmatrix} \begin{bmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{21} & \Phi_{22} \end{bmatrix} \begin{bmatrix} 0 \\ G_1 \end{bmatrix} + 0$$

$$= JC\Phi_{12}(t) G_1 + (H_2 + J D_2 H_1) \Phi_{22}(t) G_1$$

$\boxed{2}$ $m(t)\ddot{y} = -k\dot{m}(t) - m(t)g$

$$\ddot{y} = \frac{-k}{m(t)}\dot{m}(t) - g \quad , \dot{m} = u.$$

$$\begin{bmatrix} \dot{y} \\ \ddot{y} \\ \dot{m} \end{bmatrix} = \begin{bmatrix} \dot{y} \\ -g - \frac{k}{m(t)} \cdot u \\ u \end{bmatrix}$$

1. $$\dot{x} = \begin{bmatrix} x_2 \\ -g - \frac{k}{x_3} \cdot u \\ u \end{bmatrix} = f(x)$$

2. $$\left.\frac{\partial f}{\partial x}\right|_{\substack{x=x_0, \\ u=u_0}} = \left.\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & +\frac{k \cdot u}{x_3^2} \\ 0 & 0 & 0 \end{bmatrix}\right|_{\substack{x=x_0, \\ u=u_0}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & \frac{-k\epsilon}{(M-\epsilon t)^2} \\ 0 & 0 & 0 \end{bmatrix} \triangleq A(t)$$

$$\left.\frac{\partial f}{\partial u}\right|_{\substack{u=u_0, \\ x=x_0}} = \left.\begin{bmatrix} 0 \\ -\frac{k}{x_3} \\ 1 \end{bmatrix}\right|_{\substack{x=x_0, \\ u=u_0}} = \begin{bmatrix} 0 \\ \frac{-k}{(M-\epsilon t)} \\ 1 \end{bmatrix} \triangleq B(t)$$

$$\delta\dot{x} = A(t)\,\delta x(t) + B(t)\,u(t)$$

$$y(t) = \underbrace{\begin{bmatrix} 1 & 0 & 0 \end{bmatrix}}_{C(t)} \delta x(t) + \underbrace{0}_{D(t)} \cdot u(t)$$

$\boxed{3}$ (1): $A_1 = \begin{bmatrix} 2 & 1 & 2 \\ 0 & 2 & 2 \\ 0 & 0 & 1 \end{bmatrix}$ (2): $A_2 = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & -1 \end{bmatrix}$

1. $0 = \begin{vmatrix} \lambda-2 & -1 & -2 \\ & \lambda-2 & -2 \\ & & \lambda-1 \end{vmatrix} = (\lambda-2)^2(\lambda-1)$ $0 = \begin{vmatrix} \lambda-2 & -1 & -1 \\ & \lambda-2 & -1 \\ & & \lambda+1 \end{vmatrix} = (\lambda-2)^2(\lambda+1)$

eigenvalues are not the same so matrices cannot be similar.

2. All eigenvalues have negative real part
$\therefore$ asymptotic stability, which is synonymous w/ exponential stability for LTI systems.

3. $\dot{x} = Ax$, $\Delta(\lambda) = (\lambda-(2+i))(\lambda-(2-i))(\lambda-2)(\lambda-2)^3$
$= (\lambda^2 - 4\lambda + 5)(\lambda-2)^4$

4. 4 linearly independent eigenvectors:
(1 Jordan block of order 3, plus 3 non-repeated eigenvalues.)

$\boxed{14}$ $\quad x[1] = A \cdot x[0] \implies \begin{bmatrix} 4 \\ -2 \end{bmatrix} = A \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 5 \\ -2 \end{bmatrix} = A \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

2. $\therefore \begin{bmatrix} 4 & 5 \\ -2 & -2 \end{bmatrix} = A \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$

$$\begin{bmatrix} 4 & 5 \\ -2 & -2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}^{-1} = A$$

$$\begin{bmatrix} 3 & 1 \\ -2 & 0 \end{bmatrix} = A$$

1. $x[0] = \begin{bmatrix} 3 \\ -1 \end{bmatrix} = \alpha \begin{bmatrix} 4 \\ -2 \end{bmatrix} + \beta \begin{bmatrix} 5 \\ -2 \end{bmatrix}$ using linearity.

$$\begin{bmatrix} 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 3 \\ -1 \end{bmatrix} = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$$

$$\begin{bmatrix} 7 \\ -4 \end{bmatrix} = \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \implies x[1] = 7 \begin{bmatrix} 4 \\ -2 \end{bmatrix} - 4 \begin{bmatrix} 5 \\ -2 \end{bmatrix}$$

$$= \begin{bmatrix} 8 \\ -6 \end{bmatrix}$$