MECH468
Modern Control Engineering
MECH550P
Foundations in Control Engineering

Realization

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Motivation

- Now, we have learned two ways to describe LTI systems, i.e., TF and SS models.

TF: $y(s) = G(s)u(s)$ *SS:* $\begin{cases} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t) \end{cases}$

- To use analysis & design techniques for SS models, one may want to transform a TF model to an equivalent SS model. How?
- More generally, what is the relationship between two models?

From SS to TF (review)

- CT LTI SS model $\begin{cases} \frac{dx(t)}{dt} &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t) \end{cases}$

- Laplace transform with x(0)=0

$$\begin{cases} sX(s) - \cancel{x(0)} &= AX(s) + BU(s) \\ Y(s) &= CX(s) + DU(s) \end{cases}$$

$$\Rightarrow \begin{cases} X(s) &= (sI - A)^{-1}BU(s) \\ Y(s) &= CX(s) + DU(s) \end{cases}$$

Memorize this!

$$\Rightarrow Y(s) = \underbrace{\boxed{\{C(sI - A)^{-1}B + D\}}}_{=:G(s)} U(s)$$

Realization: From TF to SS

- Given a rational proper transfer matrix $G(s)$ find matrices (A,B,C,D) s.t.

$$G(s) = C(sI - A)^{-1}B + D$$

  - Rationality: (polynomial)/(polynomial)
    Rational $\frac{s+1}{s+2}$ Non-rational e^{-s}
  - Properness: deg(num)<=deg(den)
    Proper $\frac{s+1}{s+2}, \frac{1}{s+1}$ Non-proper $s+1, \frac{s^2+2s+3}{s+1}$

Remarks

- For one SS model, there is a unique TF.
- For one TF model, there are infinitely many SS!

$$(A,B,C,D) \underset{\text{infinitely many}}{\overset{\text{unique}}{\rightleftarrows}} G(s)$$

(Coordinate transformation T
Order difference)

- After realization, check the correctness by recovering the original transfer matrix.

First step for realization

- Always extract D-matrix first!

$$G(s) = \underbrace{G(\infty)}_{\text{constant}} + \underbrace{G_{sp}(s)}_{\text{Strictly proper}} \qquad \begin{aligned} G(\infty) &= D \\ G_{sp}(s) &= C(sI-A)^{-1}B \end{aligned}$$

deg (num) < deg (den)

  - Ex $\begin{bmatrix} \frac{s+2}{s+1} & \frac{1}{s+2} \\ \frac{s}{s/2+1} & 1 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}}_{\text{constant}} + \underbrace{\begin{bmatrix} \frac{1}{s+1} & \frac{1}{s+2} \\ \frac{-4}{s+2} & 0 \end{bmatrix}}_{\text{Strictly proper}}$

- After extracting D, find (A,B,C) s.t.

$$G_{sp}(s) = C(sI-A)^{-1}B$$

Controllable canonical form

- SISO example $G(s) = \dfrac{n_1 s^2 + n_2 s + n_3}{s^3 + \alpha_1 s^2 + \alpha_2 s + \alpha_3}, \; n_i \in \mathbb{R}$

$$\Rightarrow \boxed{\begin{cases} \dot{x}(t) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\alpha_3 & -\alpha_2 & -\alpha_1 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(t) \\ y(t) = \begin{bmatrix} n_3 & n_2 & n_1 \end{bmatrix} x(t) \end{cases}}$$

or equivalently $\begin{cases} \dot{x}(t) = \begin{bmatrix} -\alpha_1 & -\alpha_2 & -\alpha_3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u(t) \\ y(t) = \begin{bmatrix} n_1 & n_2 & n_3 \end{bmatrix} x(t) \end{cases}$

Companion matrix

- The following form (and its transpose) of a matrix is called *companion matrix (form):*

$$A := \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ 0 & 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \cdots & 1 \\ -\alpha_n & -\alpha_{n-1} & \cdots & -\alpha_2 & -\alpha_1 \end{bmatrix} \in \mathbb{R}^{n\times n}$$

- Important property of a companion matrix

$$\det(sI-A) = s^n + \alpha_1 s^{n-1} + \cdots + \alpha_{n-1}s + \alpha_n$$

SISO example

- Ex.1 $G(s) = \dfrac{5s^2+3s+2}{s^3+11s^2+14s+6}$

- Ex.2 $G(s) = \dfrac{4s^2+3s+2}{2s^2+5s+6}$

Controllable canonical form

- MIMO case $G(s) = \dfrac{N_1 s^{r-1} + N_2 s^{r-2} + \cdots + N_{r-1}s + N_r}{\underbrace{s^r + \alpha_1 s^{r-1} + \cdots + \alpha_{r-1}s + \alpha_r}_{\text{Least common denominator}}},\ N_i \in \mathbb{R}^{q\times p}$

$$\dot{x} = \overbrace{\begin{bmatrix} 0 & I_p & 0 & \cdots & 0 \\ 0 & 0 & I_p & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & I_p \\ -\alpha_r I_p & -\alpha_{r-1} I_p & \cdots & -\alpha_2 I_p & -\alpha_1 I_p \end{bmatrix}}^{A \in \mathbb{R}^{rp\times rp}} x + \underbrace{\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ I_p \end{bmatrix}}_{B \in \mathbb{R}^{rp\times p}} u$$

$$y = \underbrace{\begin{bmatrix} N_r & N_{r-1} & \cdots & N_2 & N_1 \end{bmatrix}}_{C \in \mathbb{R}^{q\times rp}} x$$

MIMO example

- Transfer matrix

$$\begin{aligned} G(s) &= \begin{bmatrix} \frac{1}{s^2+4s+3} & \frac{1}{s+3} \end{bmatrix} \\ &= \frac{1}{s^2+4s+3}\begin{bmatrix} 1 & s+1 \end{bmatrix} \\ &= \frac{1}{s^2+4s+3}\left\{\begin{bmatrix} 0 & 1 \end{bmatrix} s + \begin{bmatrix} 1 & 1 \end{bmatrix}\right\} \end{aligned}$$

$$\begin{cases} \dot{x} = \left[\begin{array}{cc|cc} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \hline -3 & 0 & -4 & 0 \\ 0 & -3 & 0 & -4 \end{array}\right] x + \left[\begin{array}{cc} 0 & 0 \\ 0 & 0 \\ \hline 1 & 0 \\ 0 & 1 \end{array}\right] u \\ y = \left[\begin{array}{cc|cc} 1 & 1 & 0 & 1 \end{array}\right] x \end{cases}$$

Note that the size of A-matrix is four.

Remark

- Controllable canonical realization is always controllable (but not always observable). Why?

$$\dot{x} = \begin{bmatrix} 0 & I_p & 0 & \cdots & 0 \\ 0 & 0 & I_p & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & I_p \\ -\alpha_r I_p & -\alpha_{r-1} I_p & \cdots & -\alpha_2 I_p & -\alpha_1 I_p \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ I_p \end{bmatrix} u$$

$\mathcal{C} =$

Observable canonical form

- SISO example $G(s) = \dfrac{n_1 s^2 + n_2 s + n_3}{s^3 + \alpha_1 s^2 + \alpha_2 s + \alpha_3}, \; n_i \in \mathbb{R}$

$$\boxed{\begin{cases} \dot{x}(t) &= \begin{bmatrix} 0 & 0 & -\alpha_3 \\ 1 & 0 & -\alpha_2 \\ 0 & 1 & -\alpha_1 \end{bmatrix} x(t) + \begin{bmatrix} n_3 \\ n_2 \\ n_1 \end{bmatrix} u(t) \\ y(t) &= \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} x(t) \end{cases}}$$

or equivalently
$$\begin{cases} \dot{x}(t) &= \begin{bmatrix} -\alpha_1 & 1 & 0 \\ -\alpha_2 & 0 & 1 \\ -\alpha_3 & 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} u(t) \\ y(t) &= \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} x(t) \end{cases}$$

SISO example

- Ex.1 $G(s) = \dfrac{5s^2 + 3s + 2}{s^3 + 11s^2 + 14s + 6}$

- Ex.2 $G(s) = \dfrac{4s^2 + 3s + 2}{2s^2 + 5s + 6}$

Observable canonical form

- MIMO case $G(s) = \dfrac{N_1 s^{r-1} + N_2 s^{r-2} + \cdots + N_{r-1}s + N_r}{\underbrace{s^r + \alpha_1 s^{r-1} + \cdots + \alpha_{r-1}s + \alpha_r}_{\text{Least common denominator}}}, \; N_i \in \mathbb{R}^{q \times p}$

Least common denominator

$$\boxed{\begin{aligned} \dot{x} &= \overbrace{\begin{bmatrix} O & O & \cdots & O & -\alpha_r I_q \\ I_q & O & \cdots & O & -\alpha_{r-1} I_q \\ O & \ddots & \ddots & \vdots & \vdots \\ \vdots & O & I_q & O & -\alpha_2 I_q \\ O & \cdots & O & I_q & -\alpha_1 I_q \end{bmatrix}}^{A \in \mathbb{R}^{rq \times rq}} x + \underbrace{\begin{bmatrix} N_r \\ N_{r-1} \\ \vdots \\ N_2 \\ N_1 \end{bmatrix}}_{B \in \mathbb{R}^{rq \times p}} u \\ y &= \underbrace{\begin{bmatrix} 0 & 0 & \cdots & 0 & I_q \end{bmatrix}}_{C \in \mathbb{R}^{q \times rq}} x \end{aligned}}$$

MIMO example

- Transfer matrix
$$\begin{aligned} G(s) &= \begin{bmatrix} \frac{1}{s^2+4s+3} & \frac{1}{s+3} \end{bmatrix} \\ &= \frac{1}{s^2+4s+3} \left\{ \begin{bmatrix} 0 & 1 \end{bmatrix} s + \begin{bmatrix} 1 & 1 \end{bmatrix} \right\} \end{aligned}$$

$$\begin{cases} \dot{x} = \left[\begin{array}{c|c} 0 & -3 \\ \hline 1 & -4 \end{array}\right] x + \left[\begin{array}{cc} 1 & 1 \\ \hline 0 & 1 \end{array}\right] u \\ y = \left[\begin{array}{c|c} 0 & 1 \end{array}\right] x \end{cases}$$

Note that the size of A-matrix is two, not four!

Q: What is the smallest size of A?

(Minimal realization: next lecture)

Remark

- Note the duality between controllable and observable canonical form.
- Observable canonical realization is always observable (but not always controllable). Why?

$$\dot{x} = \begin{bmatrix} O & O & \cdots & O & -\alpha_r I_q \\ I_q & O & \cdots & O & -\alpha_{r-1} I_q \\ O & \ddots & \ddots & \vdots & \vdots \\ \vdots & O & I_q & O & -\alpha_2 I_q \\ O & \cdots & O & I_q & -\alpha_1 I_q \end{bmatrix} x + \begin{bmatrix} N_r \\ N_{r-1} \\ \vdots \\ N_2 \\ N_1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 0 & 0 & \cdots & 0 & I_q \end{bmatrix} x$$

$$\Rightarrow \mathcal{O} =$$

Review

- So far, we considered the realizations of the following transfer matrix:

$$G(s) = \begin{bmatrix} \frac{1}{s^2+4s+3} & \frac{1}{s+3} \end{bmatrix} = \frac{1}{s^2+4s+3}\left\{\begin{bmatrix} 0 & 1 \end{bmatrix} s + \begin{bmatrix} 1 & 1 \end{bmatrix}\right\}$$

Controllable canonical form with order 4

$$\dot{x} = \left[\begin{array}{cc|cc} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \hline -3 & 0 & -4 & 0 \\ 0 & -3 & 0 & -4 \end{array}\right] x + \left[\begin{array}{cc} 0 & 0 \\ 0 & 0 \\ \hline 1 & 0 \\ 0 & 1 \end{array}\right] u$$

$$y = \left[\begin{array}{cc|cc} 1 & 1 & 0 & 1 \end{array}\right] x$$

Observable canonical form with order 2

$$\dot{x} = \left[\begin{array}{c|c} 0 & -3 \\ \hline 1 & -4 \end{array}\right] x + \left[\begin{array}{cc} 1 & 1 \\ \hline 0 & 1 \end{array}\right] u$$

$$y = \left[\begin{array}{c|c} 0 & 1 \end{array}\right] x$$

Topics from now on

- Realization of the smallest order
  - How to characterize such realization?
  - How to obtain such realization?
- Some terminologies:
  - *Minimal realization* of G(s):
    Realization (A,B,C,D) of G that has the smallest dimension of A-matrix.
  - *McMillan degree* of G(s):
    The dimension of A of the minimal realization. (This indicates the "complexity" of a system G(s).)

Why minimal realization?

- Easy to …
  - Analyze (understand) the system
  - Design a controller
  - Implement a controller
- Computationally less demanding in both design and implementation
- Higher reliability
  - Few parts to go wrong in the hardware
  - Few bugs to fix in the software

Two important facts on minimal realization

- *Fact 1: A realization (A,B,C,D) is minimal if and only if (A,B) is controllable and (A,C) is observable.*
- *Fact 2: All minimal realizations of G(s) are related by coordinate transformations.*

(Sketches of proofs are given in the appendix.)

Nonminimal realization examples

- Ex $G(s) = \dfrac{s^2-1}{4(s^3-1)} = \dfrac{1/4s^2 - 1/4}{s^3-1}$
  - Controllable canonical form

$$\begin{cases} \dot{x} &= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u \\ y &= [-1/4 \ 0 \ 1/4]\, x \end{cases}$$

Is it observable? ➡ No!

$$\text{rank}\begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix} = \text{rank}\begin{bmatrix} -1 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix} /4 = 2$$

  - Observable canonical form

$$\begin{cases} \dot{x} &= \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} x + \begin{bmatrix} -1/4 \\ 0 \\ 1/4 \end{bmatrix} u \\ y &= [0 \ 0 \ 1]\, x \end{cases}$$

Is it controllable? ➡ No!

$$\text{rank}\begin{bmatrix} B & AB & A^2B \end{bmatrix} = \text{rank}\begin{bmatrix} -1 & 1 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix} /4 = 2$$

How to obtain minimal realization

- SISO (Single-Input-Single-Output) case
  - Remove common factors from numerator and denominator of G(s). Then, realize G in a controllable (or observable) canonical form.

$$G(s) = \frac{s^2-1}{4(s^3-1)} = \frac{1/4s^2 - 1/4}{s^3-1} = \frac{1/4s + 1/4}{s^2+s+1}$$

➡ C.C.F. $\begin{cases} \dot{x} &= \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \\ y &= [1/4 \ 1/4]\, x \end{cases}$

➡ Observable! ➡ Minimal! McMillan degree 2

How to obtain minimal realization

- SIMO case (G(s) is a column vector)
  - Use the controllable canonical form.

$$G(s) = \frac{1}{s^2+4s+3}\left\{\begin{bmatrix} 0 \\ 1 \end{bmatrix} s + \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\} \Rightarrow \quad \dot{x} = \left[\begin{array}{c|c} 0 & 1 \\ \hline -3 & -4 \end{array}\right] x + \left[\begin{array}{c} 0 \\ \hline 1 \end{array}\right] u, \quad y = \left[\begin{array}{c|c} 1 & 0 \\ 1 & 1 \end{array}\right] x$$

- MISO case (G(s) is a row vector)
  - Use the observable canonical form.

$$G(s) = \frac{1}{s^2+4s+3}\left\{\begin{bmatrix} 0 & 1 \end{bmatrix} s + \begin{bmatrix} 1 & 1 \end{bmatrix}\right\} \Rightarrow \quad \dot{x} = \left[\begin{array}{c|c} 0 & -3 \\ \hline 1 & -4 \end{array}\right] x + \left[\begin{array}{cc} 1 & 1 \\ \hline 0 & 1 \end{array}\right] u, \quad y = \left[\begin{array}{c|c} 0 & 1 \end{array}\right] x$$

How to obtain minimal realization

- MIMO (Multiple-Input-Multiple-Output) case
  1. Realize G(s) in some non-minimal canonical form.
  2. Use the Kalman decomposition to remove uncontrollable/unobservable parts. ("minreal.m")

Remark: Unfortunately, it is generally hard to compute the Kalman decomposition by hand.

Remark: There is another famous algorithm, called *Ho's algorithm*, to compute a minimal realization. (Not covered in this course.)

Kalman decomposition (review)

- Every SS model can be transformed by z=Tx for some appropriate T into a canonical form:

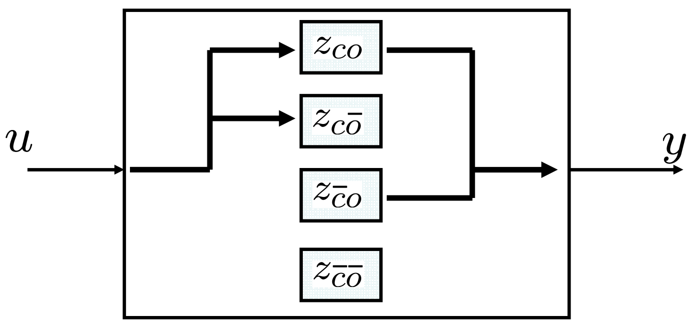
$$\begin{cases} \begin{bmatrix}\dot z_{co}\\ \dot z_{c\bar o}\\ \dot z_{\bar c o}\\ \dot z_{\bar c\bar o}\end{bmatrix} = \underbrace{\begin{bmatrix} A_{co} & 0 & A_{13} & 0\\ A_{21} & A_{c\bar o} & A_{23} & A_{24}\\ 0 & 0 & A_{\bar c o} & 0\\ 0 & 0 & A_{43} & A_{\bar c\bar o}\end{bmatrix}}_{TAT^{-1}} \begin{bmatrix} z_{co}\\ z_{c\bar o}\\ z_{\bar c o}\\ z_{\bar c\bar o}\end{bmatrix} + \underbrace{\begin{bmatrix} B_{co}\\ B_{c\bar o}\\ 0\\ 0\end{bmatrix}}_{TB} u \\ y = \underbrace{\begin{bmatrix} C_{co} & 0 & C_{\bar c o} & 0\end{bmatrix}}_{CT^{-1}} \begin{bmatrix} z_{co}\\ z_{c\bar o}\\ z_{\bar c o}\\ z_{\bar c\bar o}\end{bmatrix} + Du \end{cases}$$

$\boxed{(A_{co}, B_{co}) \text{ is controllable \& } (A_{co}, C_{co}) \text{ is observable}}$

Kalman decomposition (review)

- Conceptual figure (Not block diagram!)

System



Example

- TF $G(s) = \begin{bmatrix} \frac{4s-10}{2s+1} & \frac{3}{s+2} \\ \frac{1}{(2s+1)(s+2)} & \frac{1}{(s+2)^2}\end{bmatrix}$

```
sys11 = tf([4 -10],[2 1]);
sys12 = tf(3,[1 2]);
sys21 = tf(1,[2 5 2]);
sys22 = tf(1,[1 4 4]);
sysG = [sys11 sys12; sys21 sys22];
G = ss(sysG);
```

```
>> G

a =
```

	x1	x2	x3	x4	x5	x6
x1	-0.5	0	0	0	0	0
x2	0	-2.5	-1	0	0	0
x3	0	1	0	0	0	0
x4	0	0	0	-2	0	0
x5	0	0	0	0	-4	-2
x6	0	0	0	0	2	0

Order 6

Example (cont'd)

- Minimal realization

```
>> [Gmin,P] = minreal(G);
>> Gmin

a =
            x1          x2          x3
   x1   -0.7192      0.4738     -0.2042
   x2    0.5926      -1.781      0.5519
   x3 -9.714e-016 -6.106e-016        -2
```

Order 3

```
% Check the minimality
Ctr = ctrb(Gmin.a,Gmin.b);
Obs = obsv(Gmin.a,Gmin.c);

>> rank(Ctr)
ans =
     3
>> rank(Obs)
ans =
     3
```

Controllable!

Observable!

Minimal!

Example (cont'd)

- After realization, check the correctness by recovering the original transfer matrix.

$$G(s)=\begin{bmatrix}\frac{4s-10}{2s+1} & \frac{3}{s+2}\\ \frac{1}{(2s+1)(s+2)} & \frac{1}{(s+2)^2}\end{bmatrix}$$

```
>> zpk(Gmin)

Zero/pole/gain from input 1 to output...
      2 (s-2.5) (s+2)^2
 #1: -----------------
      (s+0.5) (s+2)^2

        0.5 (s+2)
 #2: ---------------
      (s+0.5) (s+2)^2

Zero/pole/gain from input 2 to output...
      3 (s+2) (s+0.5)
 #1: ---------------
      (s+0.5) (s+2)^2

         (s+0.5)
 #2: ---------------
      (s+0.5) (s+2)^2
```

(1,1) **(2,1)** **(1,2)** **(2,2)**

Summary

- Realization (In Matlab, ss.m)
  - Controllable canonical form
  - Observability canonical form
  - Minimal realization
- Realization for DT systems is exactly the same as that for CT systems.

Exercises

- Obtain controllable canonical realizations for the following transfer matrices. By hand-calculation, check the correctness of your answers.

$$G_1(s)=\begin{bmatrix}\frac{4s-10}{2s+1} & \frac{3}{s+2}\\ \frac{1}{(2s+1)(s+2)} & \frac{1}{(s+2)^2}\end{bmatrix}\qquad G_2(s)=\begin{bmatrix}\frac{1}{s+1} & \frac{s+3}{s+1}\\ \frac{-1}{(s+1)(s+2)} & \frac{1}{s+2}\end{bmatrix}$$

$$G_3(s)=\begin{bmatrix}\frac{1}{s+1} & \frac{1}{s+1}\end{bmatrix}\qquad G_4(s)=\begin{bmatrix}\frac{1}{s+1} & 0\\ 0 & \frac{1}{s+1}\end{bmatrix}$$

Exercises

- Obtain realizations in observable canonical forms for the following transfer matrices.

$$G_1(s) = \begin{bmatrix} \frac{4s-10}{2s+2} & \frac{3}{2s+2} \end{bmatrix} \quad G_2(s) = \begin{bmatrix} \frac{1}{s+1} & \frac{s+3}{s+1} \end{bmatrix}$$

- Using the obtained realizations above, find realizations of the following TFs.

$$G_1(s) + G_2(s) \quad \alpha G_1(s) + G_2(s)M, \ \alpha \in \mathbb{R}, M \in \mathbb{R}^{2\times 2}$$

By hand-calculation, check the correctness by recovering the original transfer matrix.

Exercises

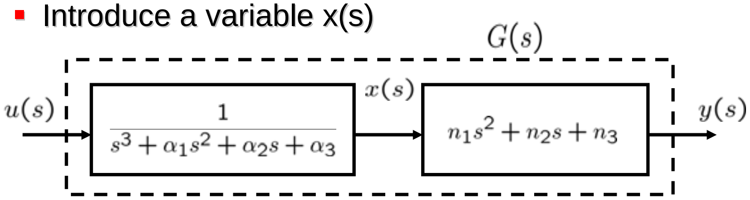
- By both hand-calculation and Matlab, obtain minimal realization for:

$$G(s) = \frac{s^3 - 5s + 2}{s^3 - 7s + 6} \qquad G(s) = \begin{bmatrix} \frac{1}{(s+1)(s+2)} & \frac{s+3}{s+2} \end{bmatrix}$$

$$G(s) = \begin{bmatrix} \frac{1}{s+1} & \frac{1}{s+1} \end{bmatrix} \qquad G(s) = \begin{bmatrix} \frac{1}{s+1} & 0 \\ 0 & \frac{1}{s+1} \end{bmatrix}$$

Appendix: Derivation of controllable canonical form

- TF $\quad G(s) = \dfrac{n_1 s^2 + n_2 s + n_3}{s^3 + \alpha_1 s^2 + \alpha_2 s + \alpha_3}, \ n_i \in \mathbb{R}$
- Introduce a variable x(s)



- We have $(s^3 + \alpha_1 s^2 + \alpha_2 s + \alpha_3) x(s) = u(s)$

$$y(s) = (n_1 s^2 + n_2 s + n_3) x(s)$$

Appendix: Derivation (cont'd)

- Introduce state variables $\begin{cases} x_1(s) := x(s) \\ x_2(s) := s x(s) \\ x_3(s) := s^2 x(s) \end{cases}$
- Then,

$$(s^3 + \alpha_1 s^2 + \alpha_2 s + \alpha_3) x(s) = u(s)$$

$$\Rightarrow \quad s x_3(s) = -\alpha_1 x_3(s) - \alpha_2 x_2(s) - \alpha_3 x_1(s) + u(s)$$

$$y(s) = (n_1 s^2 + n_2 s + n_3) x(s)$$

$$\Rightarrow \quad y(s) = n_1 x_3(s) + n_2 x_2(s) + n_3 x_1(s)$$

- By inverse Laplace transform, done!

Appendix: Derivation of observable canonical form

- TF

$$G(s)=\frac{n_1s^2+n_2s+n_3}{s^3+\alpha_1s^2+\alpha_2s+\alpha_3},\ n_i\in\mathbb{R}$$

- Rewrite I/O relation as

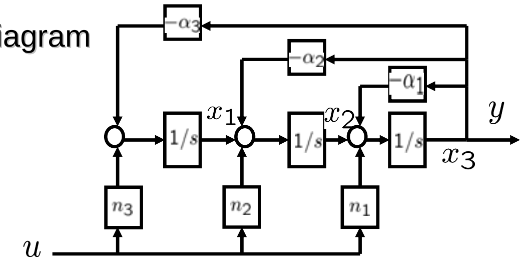
$$y(s)=\frac{n_1s^2+n_2s+n_3}{s^3+\alpha_1s^2+\alpha_2s+\alpha_3}u(s)$$

$$\Rightarrow (s^3+\alpha_1s^2+\alpha_2s+\alpha_3)y(s)=(n_1s^2+n_2s+n_3)u(s)$$

$$\Rightarrow y(s)=-\frac{\alpha_1}{s}y(s)-\frac{\alpha_2}{s^2}y(s)-\frac{\alpha_3}{s^3}y(s)+\frac{n_1}{s}u(s)+\frac{n_2}{s^2}u(s)+\frac{n_3}{s^3}u(s)$$

Appendix: Derivation (cont'd)

- Draw block-diagram



- By introducing state variables as in the figure, done!

$$\begin{cases}\dot{x}(t) &= \begin{bmatrix}0&0&-\alpha_3\\1&0&-\alpha_2\\0&1&-\alpha_1\end{bmatrix}x(t)+\begin{bmatrix}n_3\\n_2\\n_1\end{bmatrix}u(t)\\ y(t) &= \begin{bmatrix}0&0&1\end{bmatrix}x(t)\end{cases}$$

Appendix: Proof of Fact 1

- "only if"-part : If (A,B) is not controllable or (A,C) is not observable, due to Kalman decomposition theorem, there is a realization with less state dimension.
- "if"-part : Suppose (A,B,C) controllable & observable.

$$G(s)=C(sI_n-A)^{-1}B+D=\bar{C}(sI_{\bar n}-\bar{A})^{-1}\bar{B}+\bar{D},\ \bar{n}<n$$

$$\Rightarrow CA^mB=\bar{C}\bar{A}^m\bar{B},\ m=0,1,\dots$$

Contradiction!

$$\Rightarrow \mathcal{O}\mathcal{C}=\bar{\mathcal{O}}\bar{\mathcal{C}} \quad \Rightarrow \text{rank}(\mathcal{O}\mathcal{C})=\text{rank}(\bar{\mathcal{O}}\bar{\mathcal{C}}) \quad \Rightarrow n\le\bar{n}$$

$$\mathcal{O}:=\begin{bmatrix}C\\CA\\\vdots\\CA^{n-1}\end{bmatrix}$$

$$\mathcal{C}:=\begin{bmatrix}B & AB & \cdots & A^{n-1}B\end{bmatrix}$$

$P\in\mathbb{R}^{l\times m},\ Q\in\mathbb{R}^{m\times n}$

$\text{rank}P+\text{rank}Q-m\le\text{rank}(PQ)\le\min\{\text{rank}P,\text{rank}Q\}$

Appendix: Proof of Fact 2

- Given a rational proper transfer matrix, assume two minimal realizations:

$$C(sI-A)^{-1}B+D=\bar{C}(sI-\bar{A})^{-1}\bar{B}+\bar{D} \Rightarrow \mathcal{O}\mathcal{C}=\bar{\mathcal{O}}\bar{\mathcal{C}},\ \mathcal{O}A\mathcal{C}=\bar{\mathcal{O}}\bar{A}\bar{\mathcal{C}}\ \bigstar$$

Define T as $T:=(\bar{\mathcal{O}}^T\bar{\mathcal{O}})^{-1}\bar{\mathcal{O}}^T\mathcal{O}=\bar{\mathcal{C}}\mathcal{C}^T(\mathcal{C}\mathcal{C}^T)^{-1}$ (by ★)

Then, due to ★ $T^{-1}=(\mathcal{O}^T\mathcal{O})^{-1}\mathcal{O}^T\bar{\mathcal{O}}=\mathcal{C}\bar{\mathcal{C}}^T(\bar{\mathcal{C}}\bar{\mathcal{C}}^T)^{-1}$

One can verify the following by using ★

$$\bar{A}=TAT^{-1},\ \bar{B}=TB,\ \bar{C}=CT^{-1}$$