EECE 571M
Nonlinear Systems and Control
Problem Set #1

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1. Consider a mechanical model of a mass M and a spring with spring constant k, on a conveyor belt moving with constant velocity b. The force due to friction, $F_b(\dot{x})$, models "sticky" friction, or "stiction".

$$M\ddot{x} + F_b(\dot{x}) + kx = 0, \quad F_b(\dot{x}) = \begin{cases} ((\dot{x}-b)-c)^2 + d & \text{for } \dot{x} \geq b \\ -((\dot{x}-b)+c)^2 - d & \text{for } \dot{x} < b \end{cases} \tag{1}$$

   Assume that $c = 2, d = 3, M = 3, k = 3$ and the conveyer belt speed is $b > 0$.

   (a) Calculate all equilibrium points.

   (b) Determine the stability and type of the equilibrium point.

   (c) Make two phase plane plots of x vs. $\dot{x}$, using Matlab or PPLANE, for $b = 1$ and $b = 2.1$, for a few initial conditions. Describe the resultant phase plane plot in terms of the sticking and slipping behavior discussed in class. How do the two phase plane plots differ?

2. Consider the planar system

$$\begin{aligned} \dot{x}_1 &= (x_1 - x_2)(1 - x_1^2 - x_2^2) \\ \dot{x}_2 &= (x_1 + x_2)(1 - x_1^2 - x_2^2) \end{aligned} \tag{2}$$

   (a) Determine the stability of (0,0), the only *isolated* equilibrium point of the system.

   (b) For the linear system that approximates dynamics around the origin, find the coordinate transformation into real Jordan form.

   (c) Sketch the phase portrait and discuss the qualitative behavior of the system. Does the behavior of the linear approximation match the behavior of the nonlinear system near the origin? Why or why not?

   (d) Does this system have a periodic orbit? Why or why not?

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PS#1 SOLUTIONS

Sept. 2010

1. $M\ddot{x} + F_b(\dot{x}) + kx = 0$

$$x_1 = x, \quad x_2 = \dot{x} \implies \dot{x}_1 = x_2, \quad \dot{x}_2 = \frac{1}{M}\left(-F_b(x_2) - kx_1\right)$$

a) Equilibrium pts: $\dot{x} = f(x^*) = 0$

$x_2^* = 0$

$-\frac{1}{M}\left(F_b(x_2^*) + kx_1^*\right) = 0$

$F_b(0) = -kx_1^*$

$= -((0-b)+c)^2 - d$

$= -(-b+c)^2 - d$

$x_1^* = +\dfrac{(-b+c)^2 + d}{k}$

$= \dfrac{(-b+2)^2 + 3}{3}$

b) $D_f = \begin{bmatrix} 0 & 1 \\ -\frac{k}{M} & -\frac{\partial F_b/\partial x_2}{M} \end{bmatrix}\Bigg|_{x^*} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{M} & +\frac{1}{M}\cdot 2((x_2-b)+c) \end{bmatrix}\Bigg|_{x^*}$

$= \begin{bmatrix} 0 & 1 \\ -\frac{k}{M} & \frac{2(c-b)}{M} \end{bmatrix} \implies 0 = \begin{vmatrix} \lambda & -1 \\ k/M & \lambda - 2\frac{(c-b)}{M} \end{vmatrix} = \lambda^2 - \underbrace{2\lambda\left(\frac{c-b}{M}\right)}_{+ \text{ for } c<b} + \frac{k}{M}$

$\lambda_{1,2} = \left| \dfrac{+\frac{2(c-b)}{M} \pm \sqrt{\left(\frac{2(c-b)}{M}\right)^2 - 4\frac{k}{M}}}{2} \right.$

$$D_f = \begin{bmatrix} 0 & 1 \\ -1 & \frac{2(2-b)}{3} \end{bmatrix}$$

$$0 = \begin{vmatrix} \lambda & -1 \\ 1 & \lambda + \frac{2(b-2)}{3} \end{vmatrix} = \lambda^2 + \underbrace{\lambda \cdot \frac{2(b-2)}{3}}_{\text{stable for } b > 2} + 1$$

$$\lambda = \frac{-\frac{2(b-2)}{3} \pm \sqrt{\left(\frac{2(b-2)}{3}\right)^2 - 4 \cdot 1}}{2}$$

$$= -\frac{(b-2)}{3} \pm \sqrt{\frac{(b-2)^2}{9} - 1}$$

If $\frac{(b-2)^2}{9} < 1$, will have complex $\lambda_{1,2}$

$$(b-2) < 3$$

$$b < 5$$

$\therefore$ for

$0 < b < 2$	eq. pt is unstable focus
$b = 2$	center
$2 < b < 5$	stable focus
$b = 5$	improper node
$5 < b$	stable node

For $b = 5$, $D_f = \begin{bmatrix} 0 & -1 \\ 1 & -2 \end{bmatrix}$, $\lambda_{1,2} = -1$ and $(-1)\begin{bmatrix} v_{11} \\ v_{12} \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} v_{11} \\ v_{12} \end{bmatrix}$ since $\lambda_1 v_1 = A v_1$

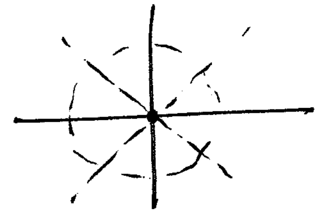
$$-v_{11} = -v_{12} \implies v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$-v_{12} = v_{11} - v_{12} \cdot 2$$

Since only 1 independent eigenvector,
x^* is improper node.

12.

$$\dot{x}_1 = (x_1 - x_2)(1 - x_1^2 - x_2^2)$$
$$\dot{x}_2 = (x_1 + x_2)(1 - x_1^2 - x_2^2)$$



a) $x^* = (0,0)$ is only isolated eq. pt

since $x = (\pm 1/\sqrt{2}, \pm 1/\sqrt{2})$ lies on $\underbrace{x_1^2 + x_2^2 = 1}$

not isolated, although $\dot{x} = 0$ for pts on unit circle.

$$D_f = \left.\begin{bmatrix} 1 - x_1^2 - x_2^2 - 2x_1(x_1 - x_2) & -(1 - x_1^2 - x_2^2) - 2x_2(x_1 - x_2) \\ 1 - x_1^2 - x_2^2 - 2x_1(x_1 + x_2) & (1 - x_1^2 - x_2^2) - 2x_2(x_1 + x_2) \end{bmatrix}\right|_{x^*}$$

$$= \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

$$0 = \begin{vmatrix} \lambda - 1 & +1 \\ -1 & \lambda - 1 \end{vmatrix} = (\lambda - 1)^2 + 1 = \lambda^2 - 2\lambda + 2$$

$$\therefore \lambda = \frac{+2 \pm \sqrt{(-2)^2 - 4 \cdot 2}}{2} = 1 \pm j,$$ unstable focus.

b) Find eigenvectors $w_{1,2} = u \pm vj$ for $\lambda = \alpha \pm \beta j = 1 \pm j$

$$A w_{1,2} = \lambda_{1,2} w_{1,2} \Rightarrow \begin{cases} Au = \alpha u - \beta v \\ Av = \beta u + \alpha v \end{cases}$$ for $A = D_f$

OR:
$$\begin{cases} 0 = \alpha \cdot I \cdot u - \beta \cdot I v - Au \\ 0 = \beta \cdot I u + \alpha \cdot I \cdot v - Av \end{cases}$$

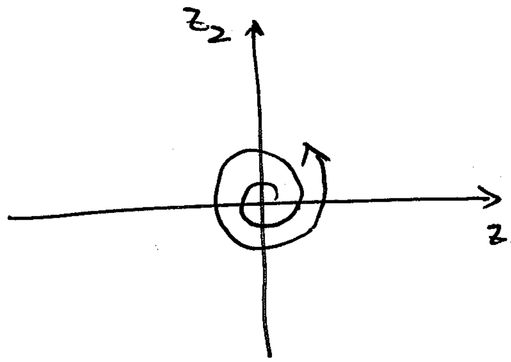
$$0 = \begin{bmatrix} \alpha I - A & -\beta I \\ \beta I & \alpha I - A \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}$$

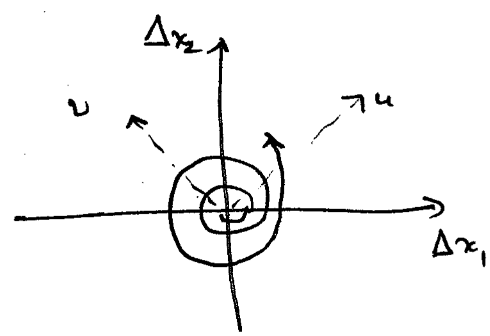
$$0 = \begin{bmatrix} 0 & 1 & -1 & 0 \\ -1 & 0 & 0 & -1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ v_1 \\ v_2 \end{bmatrix}$$

this is a reduced rank matrix, rank = 2.

$$\Rightarrow \begin{aligned} u_2 &= v_1 \\ u_1 &= -v_2 \end{aligned}$$

Choose $u = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $v = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ $\therefore$ $T = \begin{bmatrix} u \,|\, v \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$





c) Consider the transformation into polar coordinates:

$$r = (x_1^2 + x_2^2)^{1/2}$$

$$\phi = \tan^{-1}(x_2/x_1)$$

then $\dot{r} = \frac{1}{2}(x_1^2+x_2^2)^{-1/2} \cdot (2x_1\dot{x}_1 + 2x_2\dot{x}_2)$

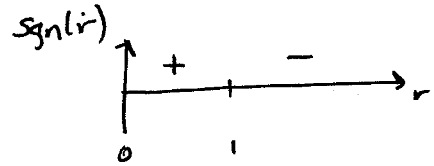
$$= \frac{x_1\dot{x}_1 + x_2\dot{x}_2}{r}$$

$$= \frac{1}{r}\left(x_1(x_1-x_2)(1-x_1^2-x_2^2) + x_2(x_1+x_2)(1-x_1^2-x_2^2)\right)$$

$$\dot{r} = \frac{1}{r}\left((1-r^2)\left(x_1^2 - x_1 x_2 + x_1 x_2 + x_2^2 \right)\right)$$

$$= \frac{1-r^2}{r} \cdot r^2$$

$$= r(1-r)$$



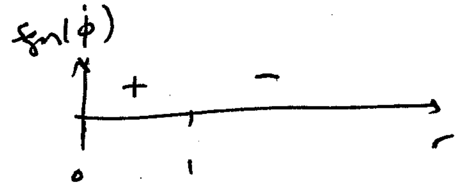
And $$\dot{\phi} = \frac{1}{1+(x_2/x_1)^2} \cdot \frac{d}{dt}(x_2/x_1)$$

$$= \frac{x_1^2}{x_1^2+x_2^2}\left(\frac{-x_2\dot{x}_1 + x_1\dot{x}_2}{x_1^2} \right)$$

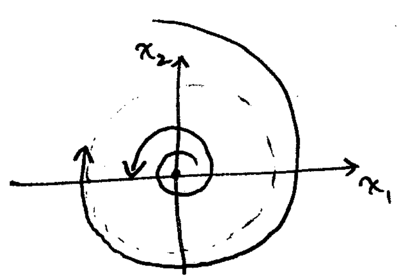
$$= \frac{1}{r^2}\left(-x_2(x_1-x_2)(1-r^2) + x_1(x_1+x_2)(1-r^2) \right)$$

$$= \frac{1-r^2}{r^2}\left(+x_2^2 + x_1^2 \right)$$

$$= 1-r^2$$



Therefore trajectories spiral towards unit circle in CW fashion for $r>1$ and in a CCW fashion for $r<1$.



Behavior near eq. pt $x^* = (0,0)$ DOES match behavior of linear approximation since Hartman-Grobman Theorem can be applied ($Re(\lambda) \neq 0$).

d) No periodic orbit exists.

Trajectories on the unit circle are frozen, hence they are not points for which $x(t+T) = x(t)$ for a <u>finite</u> $T > 0$.