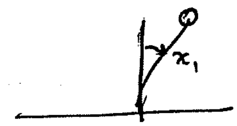
Problem Set #2 | Nonlinear Systems | M. [illegible] | Oct 2010

$\boxed{11}$

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = \frac{g}{L}\sin x_1 - \frac{b}{mL^2}x_2$$



a) $\dot{x}^* = 0$ for $x_2^* = 0$, $\sin x_1^* = 0$

$$\Rightarrow x^* = (n\pi, 0), \ n \in \mathbb{Z}.$$

nullclines: $x_2 = 0$

$$\frac{g}{L}\sin x_1 - \frac{b}{mL^2}x_2 = 0$$

b) $$D_f = \begin{bmatrix} 0 & 1 \\ \frac{g}{L}\cos x_1 & -\frac{b}{mL^2} \end{bmatrix}\Bigg|_{x^*} = \begin{bmatrix} 0 & 1 \\ \frac{g}{L}(-1)^n & -\frac{b}{mL^2} \end{bmatrix}\Bigg|_{x^* = n\pi}$$

$$0 = |\lambda I - D_f|$$
$$= \begin{vmatrix} \lambda & -1 \\ -\frac{g}{L}(-1)^n & \lambda + \frac{b}{mL^2} \end{vmatrix}$$
$$= \lambda^2 + \underbrace{\frac{b}{mL^2}}_{>0}\lambda - \underbrace{\frac{g}{L}(-1)^n}_{>0 \text{ for } n \text{ odd.}}$$

$$\lambda = \frac{-\frac{b}{mL^2} \pm \sqrt{\left(\frac{b}{mL^2}\right)^2 + 4\left(\frac{g}{L}\right)(-1)^n}}{2}$$ ← always positive for n even.

$\therefore$ focus for $\left(\frac{b}{mL^2}\right)^2 + 4\frac{g}{L}(-1)^n < 0$, n odd

$$\left(\frac{b}{mL^2}\right)^2 < 4\frac{g}{L}$$
$$\frac{b}{mL^2} < 2\sqrt{g/L}$$

For $x^* = (n\pi, 0)$, n even (inverted): saddle.

n odd (hanging): stable node for $\frac{b}{mL^2} > 2\sqrt{\frac{g}{L}}$

focus $\frac{b}{mL^2} < 2\sqrt{\frac{g}{L}}$

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Depending on the amount of damping in the joint, for equilibria that coincide w/ the pendulum hanging downwards, movement near vertical will be either overdamped (stable node) or underdamped (stable focus).

When $\frac{b}{mL^2} = 2\sqrt{g/L}$, repeated root results. Eigenvectors of Df must be analyzed to determine if x^* is an improper stable node or proper stable node.

Let $a = \frac{b}{mL^2} = 2\sqrt{\frac{g}{L}} \implies \frac{g}{L} = \left(\frac{a}{2}\right)^2$

$$D_f = \begin{bmatrix} 0 & 1 \\ -\left(\frac{a}{2}\right)^2 & -a \end{bmatrix}$$

$$0 = \lambda^2 + a\lambda + \left(\frac{a}{2}\right)^2 \implies \lambda = -\frac{a}{2}, -\frac{a}{2}$$

$$D_f \cdot \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = (-a/2)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

$$\begin{bmatrix} v_2 \\ -\left(\frac{a}{2}\right)^2 v_1 - a v_2 \end{bmatrix} = \begin{bmatrix} -a/2 \, v_1 \\ -a/2 \, v_2 \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -a/2 \, v_1 - v_2 \\ +(a/2)^2 v_1 + a/2 \cdot v_2 \end{bmatrix}$$

Choose $v_2 = -a/2 \cdot v_1 \implies \begin{bmatrix} 1 \\ -a/2 \end{bmatrix}$

But need generalized eigenvector for 2nd eigvect.

$\implies$ improper stable node for $\frac{b}{mL^2} = 2\sqrt{g/L}$

c) Bendixson's:

$$\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} = 0 - \frac{b}{mL^2} = -\frac{b}{mL^2} < 0 \quad \text{for } x \in \mathbb{R}^2$$

$\therefore$ No periodic orbits can exist in $\mathbb{R}^2$.

$\boxed{2}$ $\dot{x} = b(c(x) - y)$ $c(x) = -x^3 + 6x^2 - 9x + 6$

$\dot{y} = \frac{1}{b}(x - d^{-1}(y))$ $d(x) = x^2 \cdot \mathrm{sgn}(x)$

a) At least 1 eq. pt must occur in $x \in [1, 3]$ by Poincare Theorem.

At most 3 eq. pt could exist since $c(x)$ is a cubic.

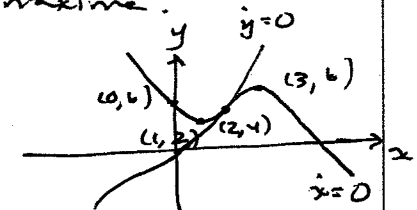
b) $\frac{\partial}{\partial x} c(x) = -3x^2 + 12x - 9 \implies 0 = x^2 - 4x + 3 = (x-1)(x-3)$

$\frac{\partial^2}{\partial x^2} c(x) = -6x + 12$

$\therefore$ $x = 1$ is local minima

$x = 3$ is local maxima.

Nullclines are not affected by value of b, although phase plane is affected.



c) $\begin{matrix}\dot{x} = 0 \\ \dot{y} = 0\end{matrix} \implies \begin{matrix} c(x^*) = y^* \\ x^* = d^{-1}(y^*)\end{matrix} \implies d(x^*) = y$

$\therefore$ $c(x^*) = d(x^*)$

$-x^{*3} + 6x^{*2} - 9x^* + 6 = x^{*2}$

$-x^{*3} + 5x^{*2} - 9x^* + 6 = 0$ $\implies x^* = 2.$

d) $D_f = \begin{bmatrix} b \cdot \frac{\partial c}{\partial x}\big|_{x^*} & -b \\ 1/b & -1/b \, \frac{\partial}{\partial y} d^{-1}(y)\big|_{y^*} \end{bmatrix}$

Notice that $\frac{\partial c}{\partial x}\Big|_{x^*} = -3(x^* - 1)(x^* - 3)$

and $\frac{\partial}{\partial y}(d^{-1}(y))\Big|_{y^*} = \frac{\partial}{\partial y}\sqrt{y}\Big|_{y^*} = \frac{1}{2\sqrt{y^*}}$

Since $d(x^*) = x^{*2} = y^*$ for $x^* > 0$,

$\frac{\partial}{\partial y}(d^{-1}(y))\Big|_{y^*} = \frac{1}{2x^*}$

D_f has eigvals:

$$0 = \begin{vmatrix} \lambda + 3b(x^*-1)(x^*-3) & b \\ -1/b & \lambda + \frac{1}{2bx^*} \end{vmatrix}$$

$$= \lambda^2 + \lambda\left(3b(x^*-1)(x^*-3) + \frac{1}{2bx^*} \right) + \frac{3}{2}\frac{(x^*-1)(x^*-3)}{x^*} + 1$$

$$= \lambda^2 + \lambda\Big(\underbrace{-3b + \frac{1}{4b}}_{>0 \text{ for } -12b^2+1>0} \Big) + \frac{1}{4}$$

STABLE for: $\quad b < \frac{1}{2\sqrt{3}}$

$$\lambda = \frac{-\left(\frac{1}{4b} - 3b\right)^2 \pm \sqrt{\left(\frac{1}{4b}-3b\right)^2 - 4\cdot\frac{1}{4}}}{2}$$

focus when $\left(\frac{1}{4b} - 3b\right)^2 - 1 < 0$

$3b - \frac{1}{4b} < 1$, OR $\quad \frac{1}{4b} - 3b < 1$

$0 < -b^2 + \frac{b}{3} + 1/12 \qquad 0 < b^2 + \frac{1}{3}b - \frac{1}{12}$

$< \underbrace{(1/2 - b)(1/6 + b)}_{\text{true for } b < 1/2} \qquad 0 < \underbrace{(b - \frac{1}{6})(b + 1/2)}_{\text{true for } b > 1/6}.$

$0 < b < 1/6$	stable node
$b = 1/6$	stable improper node
$1/6 < b < 1/2\sqrt{3}$	stable focus
$b = 1/2\sqrt{3}$	center
$\frac{1}{2\sqrt{3}} < b < 1/2$	unstable focus
$b = 1/2$	unstable improper node
$1/2 < b$	unstable node

See next page.

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When $b = 1/6$, $D_f = \begin{bmatrix} 3b & -b \\ 1/b & -1/4b \end{bmatrix} = \begin{bmatrix} 1/2 & -1/6 \\ 6 & -3/2 \end{bmatrix}$

$\Longrightarrow$ repeated eigvals w/ dependent eigvectors.

When $b = \frac{1}{2\sqrt{3}}$, $D_f = \begin{bmatrix} \sqrt{3}/2 & -\frac{1}{2\sqrt{3}} \\ 2\sqrt{3} & -\sqrt{3}/2 \end{bmatrix}$

$\longrightarrow$ eigvals on $j\omega$ axis

When $b = 1/2$, $D_f = \begin{bmatrix} 3/2 & -1/2 \\ 2 & -1/2 \end{bmatrix}$

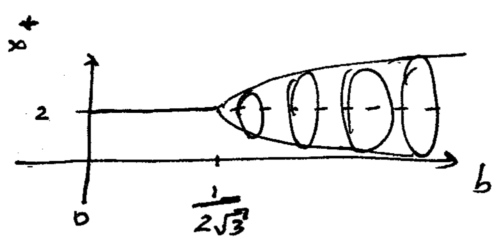
$\Longrightarrow$ repeated eigvals w/ dependent eigvectors

e) See plots next pages.

$b = 0.1$	stable node
$b = 0.3$	unstable focus w/ limit cycle
$b = 1.0$	unstable node w/ limit cycle.

f) Notice that x^* is not a function of $\mu = b$, but D_f is.

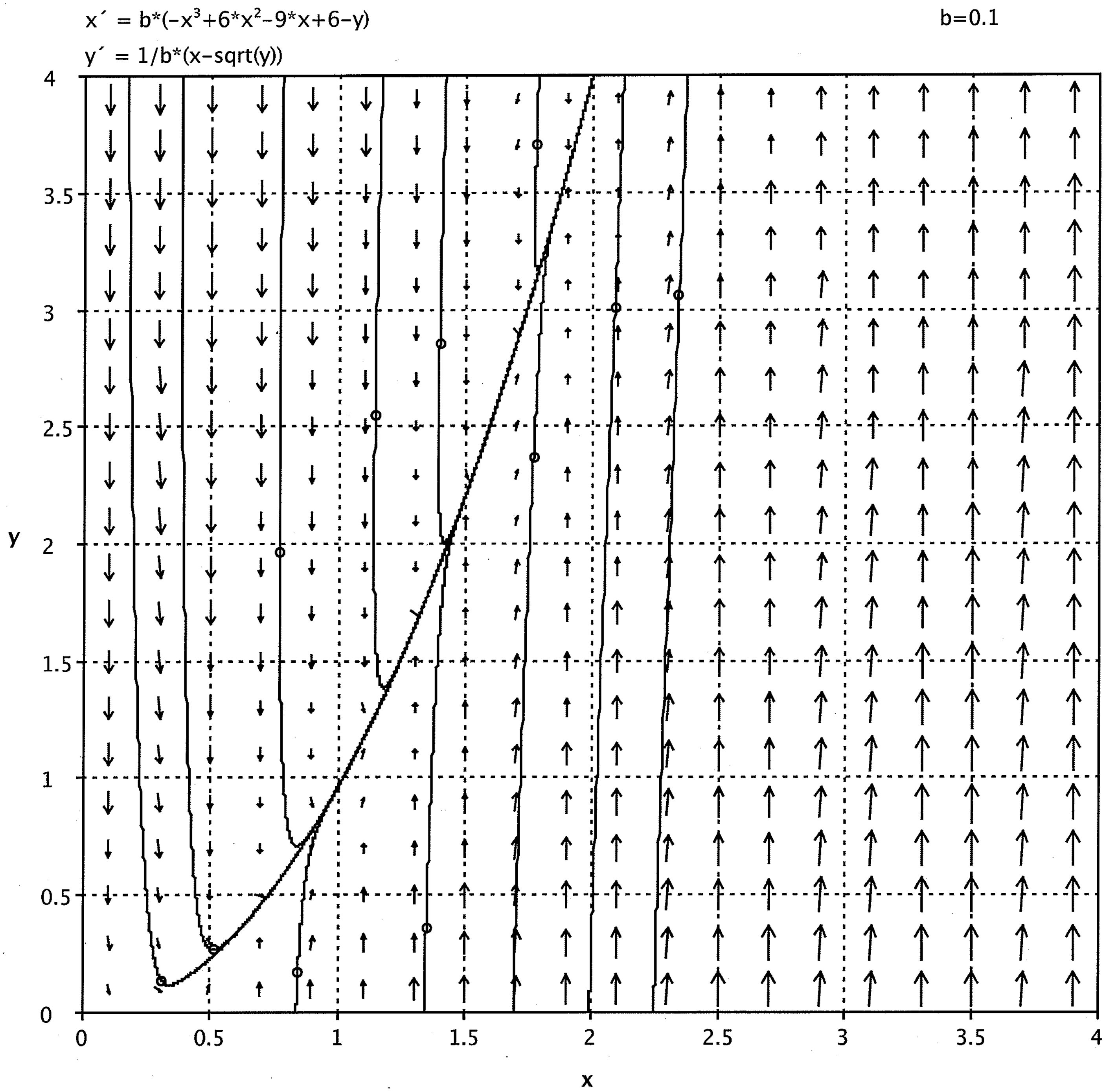
$$D_x f_\mu = \left.\frac{\partial f_\mu}{\partial x}\right|_{x^*} = \begin{bmatrix} 3b & -b \\ 1/b & -1/(4b) \end{bmatrix} \quad \text{with } \mu = b.$$

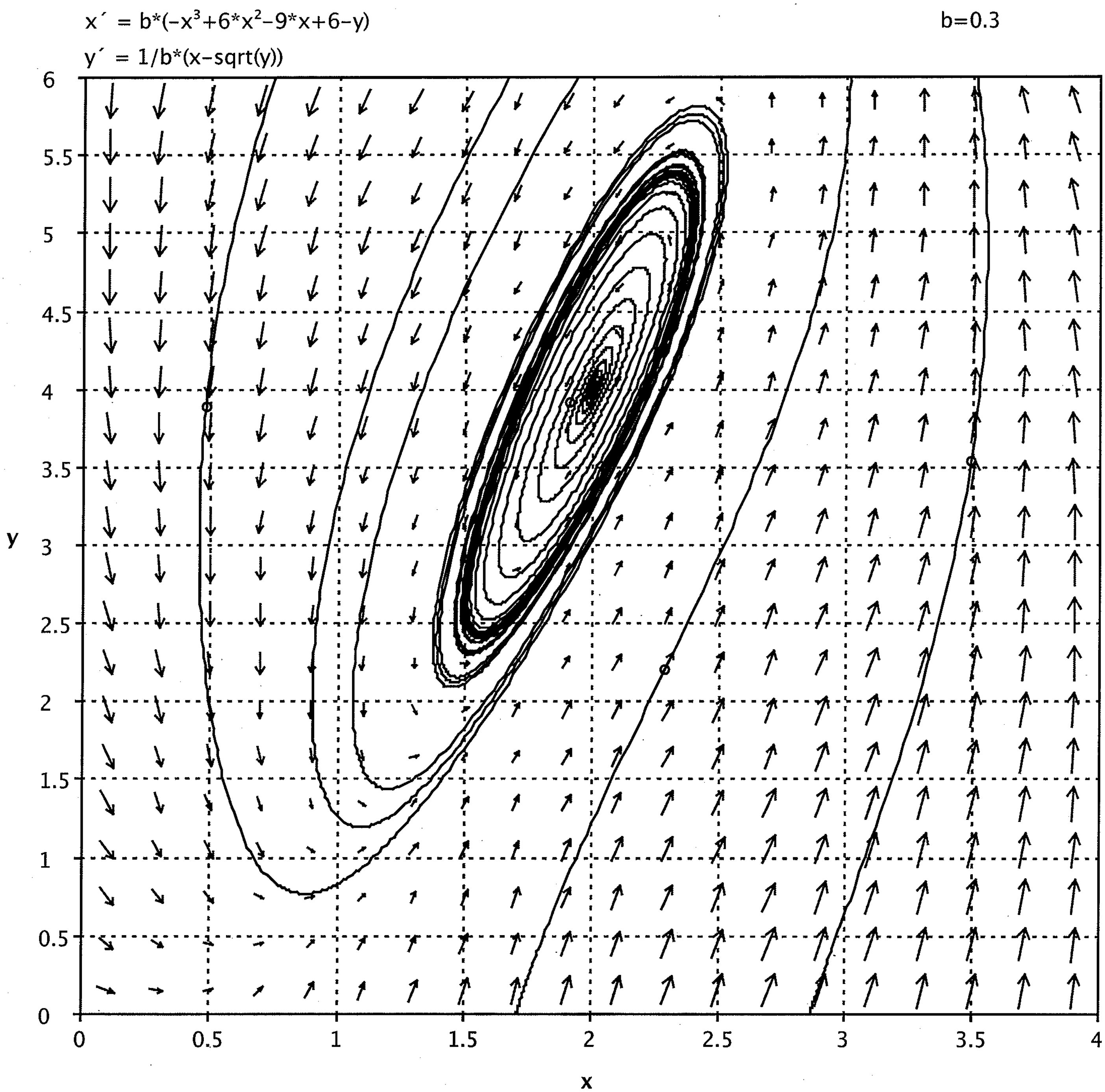


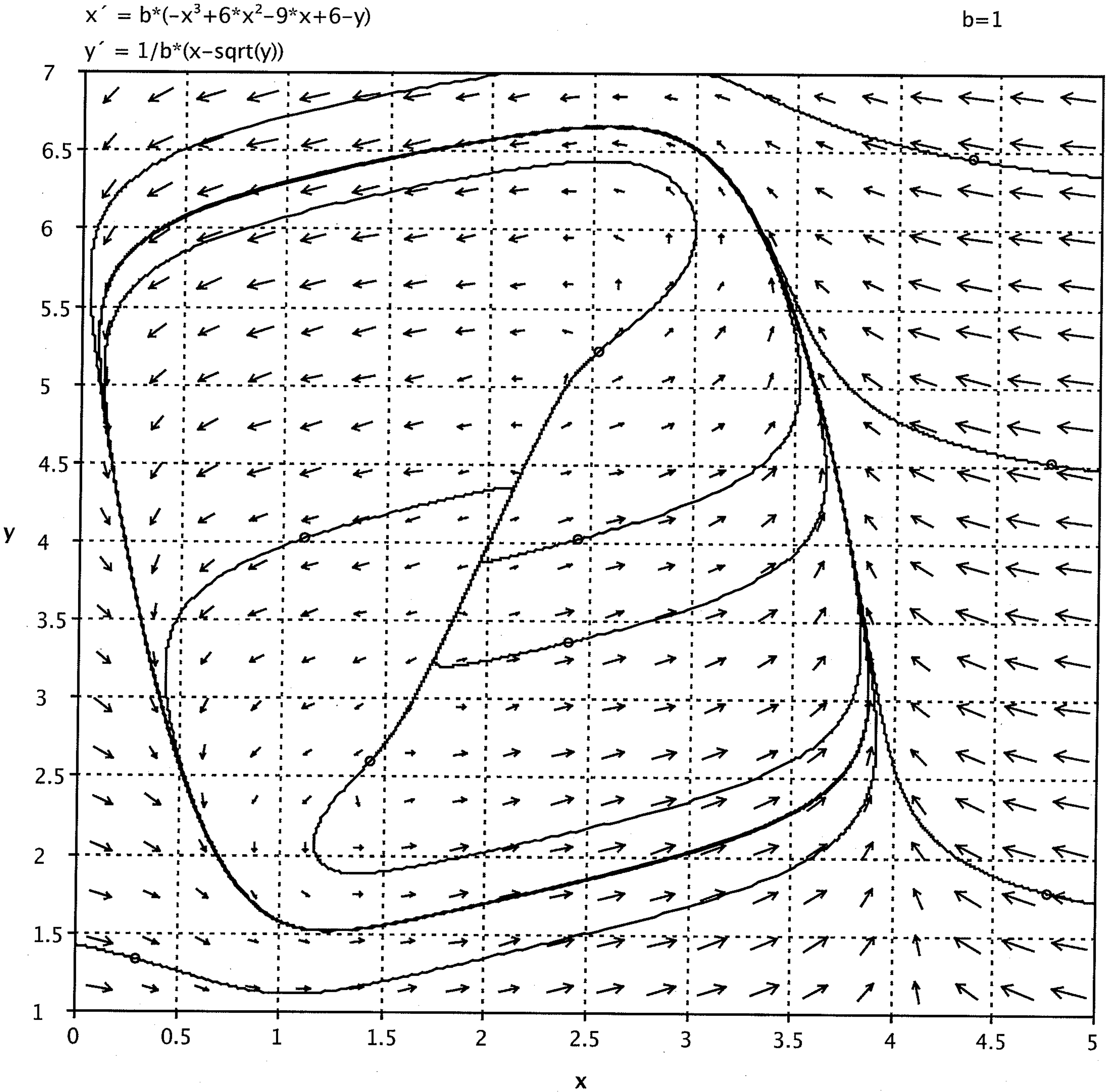
Analysis of bifurcations does not consider changes in type unless it involves a change in stability. Hence although the equilibrium type involves several changes, only 1 bifurcation occurs, at $b = \frac{1}{2\sqrt{3}}$

Based on simulation in pplane + analysis of other cubic systems (e.g. Van der Pol oscillator) it appears that a limit cycle appears for $b > \frac{1}{2\sqrt{3}}$, although further analysis must be done to prove its existence (e.g., application of Poincare-Bendixson). $\Longrightarrow$ supercritical Hopf bifurcation

g) See analysis in c).


x´ = b*(−x³+6*x²−9*x+6−y)
y´ = 1/b*(x−sqrt(y))
b=0.1
y
x
0
0.5
1
1.5
2
2.5
3
3.5
4


x´ = b*(-x³+6*x²-9*x+6-y)
y´ = 1/b*(x-sqrt(y))
b=0.3
y
x
0
0.5
1
1.5
2
2.5
3
3.5
4
4.5
5
5.5
6



x´ = b*(-x³+6*x²-9*x+6-y)
y´ = 1/b*(x-sqrt(y))
b=1
y
x
7
6.5
6
5.5
5
4.5
4
3.5
3
2.5
2
1.5
1
0
0.5
1
1.5
2
2.5
3
3.5
4
4.5
5