EECE 571M

Nonlinear Systems and Control

Dr. Meeko Oishi

Electrical and Computer Engineering

University of British Columbia, BC

http://courses.ece.ubc.ca/571m

moishi@ece.ubc.ca

Outline

- Introduction
  - Why nonlinear systems
  - Motivating examples
  - Some types of nonlinear phenomena
- Review
  - Linear systems
  - Equilibria
  - Jacobian linearization
  - Phase-plane

Learning Objectives

At the end of this lecture, you should be able to…

- Introduction
  - Distinguish linear systems from nonlinear systems mathematically
  - Qualitatively describe some phenomena particular to nonlinear systems
- Review
  - Linearize a nonlinear system about an equilibrium point
  - Relate behavior near an equilibrium point to eigenvalues of Jacobian matrix
  - Relate phase-plane plots and transient behavior for 2^{nd} order systems

References

- Khalil
  - Section 1.1,
  - Section 2.1, 2.2, 2.3
- Sastry
  - Section 1.1,1.2, 1.5
  - Section 2.1,2.2

Linear vs. Nonlinear

- Linear systems
  - $\frac{dx(t)}{dt} = A(t)x(t), \quad x(t_0) = x_0 \text{ with } x \in \Re^n, A(t) \in \Re^{n \times n}$
  - Wealth of developed tools for analysis, simulation, and control
  - Describes some behaviors of physical systems in a certain "neighborhood"
- Nonlinear systems
  - Most physical systems
  - Exhibits behaviors simply NOT POSSIBLE in linear systems

Simulation vs. Analysis

- Simulation
  - Enabled through continued advances in computing
    - Parallel processing
    - Grid computing
    - Algorithms for scientific computing
  - Coupled with good intuition, can predict system behavior

Simulation vs. Analysis

- Drawbacks to simulation
  - May be inaccurate for nonlinear systems
    - Non-Lipschitz dynamics
    - "Stiff" ODEs
    - Logic-based dynamics
    - Chattering
  - Can be highly dependent on particular choice of initial conditions
  - May be an unreliable predictor of system behavior if critical test cases are missed
  - *No proofs are possible* through simulation for stability or reachability

Simulation vs. Analysis

- Analysis
  - Mathematical proofs of system behavior
    - Stability
    - Reachability
    - Optimality
    - Robustness
    - Others…
  - Independent of choice of initial conditions (within a neighborhood)
  - May yield non-intuitive results (e.g., conditions that may not have been considered otherwise for simulation)

Simulation vs. Analysis

- Drawbacks to analysis
  - Requires analytic formulation of system dynamics (e.g., no look-up tables)
    - Ordinary differential equations

$$\frac{dx(t)}{dt} = f(x(t), t), \quad x(t_0) = x_0, \; x \in \Re^n$$

    - Difference equations

$$x(k+1) = f(x(k)), \quad x(0) = x_0, \; x(k) \in \Re^n$$

  - Computational tools for analysis are limited

Simulation vs. Analysis

- Computational tools for simulation
  - pplane
  - Matlab's ode23, ode45, etc.
  - Mathematica integration routines
  - SPICE, Xyce, etc.
- Computational tools for analysis
  - SOStools
  - Symbolic manipulation in Mathematical, Matlab symbolic toolbox
  - Reachability tools

Practical examples

- Nonlinear analysis
  - "Flutter" in aircraft wing structural dynamics
  - Oscillations in nonlinear circuits
  - Actuator hysteresis
  - Population dynamics -- Atlantic cod collapse
  - Human heart arrythmia
  - Circadian clock
- Nonlinear control
  - Helicopter flight control systems
  - VTOL and fighter aircraft FMS
  - Robot manipulation
  - Automotive fuel injection systems
  - Biochemical reactors

Equilibrium Points

<u>Definition</u>: Equilibrium point

- A point $x^* \in \mathbb{R}^n$ is an equilibrium point of

$$\dot{x} = f(x)$$

iff $f(x^*) = 0$.

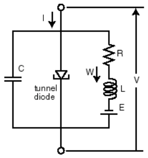
- Finding and analyzing equilibria is key to predicting the behavior of nonlinear systems

Ex. #1: Heart rhythms

- Fitzhugh-Nagumo model of electric pulses in the heart

$$\dot{V} = \frac{1}{\epsilon}(V - V^3/3 - W)$$
$$\dot{W} = \epsilon(V - \gamma W + \beta)$$

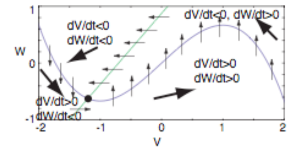
- Heart tissue is excitable -- a small impulse can trigger a large response
- Based on circuit models used to explain neuronal impulses

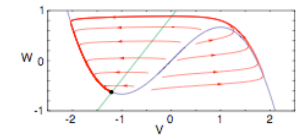


- References:
  - http://www.scholarpedia.org/article/FitzHugh-Nagumo_model
  - http://thevirtualheart.org/java/fhnphase.html

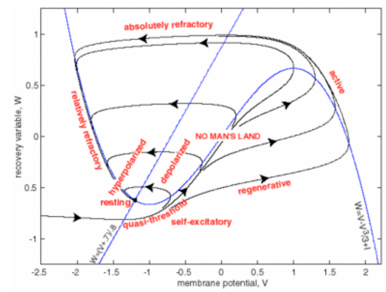
Ex. #1: Heart rhythms

- Null clines intersect at different points as β varies



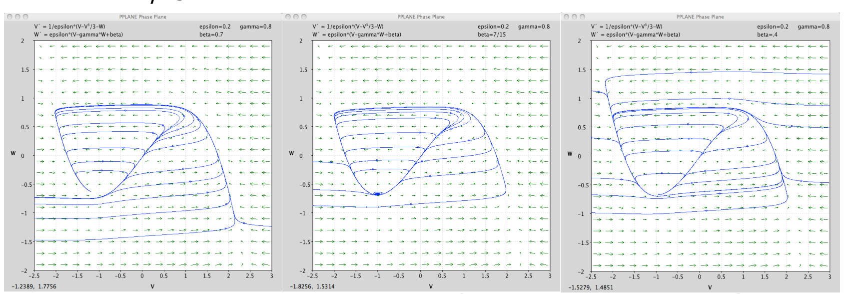


- Intersection is an equilibrium point, stable for β = 0.8
- What does this mean physically?



Ex. #1: Heart rhythms

- What happens to the stability of the equilibrium point when β < 7/15 ?



β > 7/15 β = 7/15 β < 7/15

- This phenomena is known as a **Hopf bifurcation**.

Ex. #1: Heart rhythms

- Notice that although the equilibrium point is unstable for the lowest value of β, periodic motion is achieved.
- While not 'stable' according to our previous definitions, trajectories converge to a particular path in the state-space -- this is a **limit cycle**.
- This phenomena is unique to nonlinear systems.
- But how do we know whether limit cycles exist?

Preliminaries

Some behaviors particular to **nonlinear** systems (not possible in linear systems)

- Multiple isolated equilibrium points
- Limit cycles
- Bifurcations
- Subharmonic, harmonic, or almost-periodic oscillations
- Complex dynamical behavior (chaos)

Preliminaries

Nonlinear State Model $\dot{x} = f(t, x, u)$

$$\begin{aligned} \dot{x}_1 &= f_1(t, x_1, \ldots, x_n, u_1, \ldots, u_p) \\ \dot{x}_2 &= f_2(t, x_1, \ldots, x_n, u_1, \ldots, u_p) \\ \vdots &\quad \vdots \\ \dot{x}_n &= f_n(t, x_1, \ldots, x_n, u_1, \ldots, u_p) \end{aligned}$$

$\dot{x}_i$ denotes the derivative of x_i with respect to the time variable t

$x_1, x_2, \ldots, x_n$ are state variables $x \in \mathbb{R}^n$

$u_1, u_2, \ldots, u_p$ are input variables $u \in \mathbb{R}^p$

and the system dynamics are $f : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^p \rightarrow \mathbb{R}^n$

Preliminaries

Common forms

Autonomous System:

$$\dot{x} = f(x)$$

Time-Invariant System:

$$\dot{x} = f(x, u)$$

Preliminaries

What can go wrong with generic nonlinear DEs:

- Lack of existence of solutions

$$\dot{x} = \text{sign}(x),\ x(0) = 0$$

- Lack of uniqueness of solutions

$$\dot{x} = 3x^{2/3},\ x(0) = 0$$

- Finite escape time

$$\dot{x} = 1 + x^2,\ x(0) = 0$$

These problems can be addressed by constraining *f(x)* (e.g., Lipschitz continuity)

Ex. #2: Lotka-Volterra comp.

- Competition between species (e.g., finite shared resources)

$$\dot{n}_1 = r_1 n_1 \left(1 - \frac{n_1}{K_1} - b_{12}\frac{n_2}{K_1}\right)$$
$$\dot{n}_2 = r_2 n_2 \left(1 - \frac{n_2}{K_2} - b_{21}\frac{n_1}{K_2}\right)$$

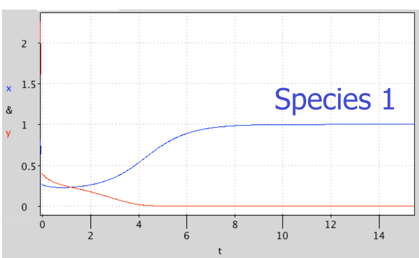
- Population of species n_1, n_2
- Constant parameters
  - Growth rates r_1,r_2
  - Carrying capacities K_1, K_2
  - Competition coefficients b_{12}, b_{21}
- Resources:
  - http://fisher.forestry.uga.edu/popdyn/LotkaVolterraCompetition.html
  - http://www.tiem.utk.edu/bioed/bealsmodules/competition.html

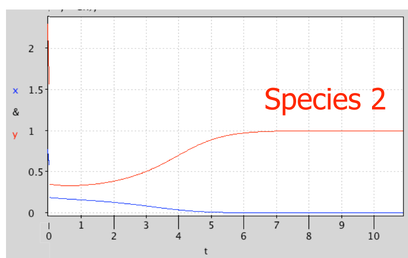
Ex. #2: Lotka-Volterra comp.

- Non-dimensionalize
  - Define new variables $x_1 = n_1/K_1$, $x_2 = n_2/K_2$
  - Define constant parameters $A = b_{12} K_1/K_2$, $B = r_2/r_1$, and $C = b_{21} K_2/K_1$
- "Normalized" dynamics

$$\dot{x}_1 = x_1 (1 - x_1 - Ax_2)$$
$$\dot{x}_2 = x_2 B (1 - x_2 - Cx_1)$$

Ex. #2: Lotka-Volterra comp.

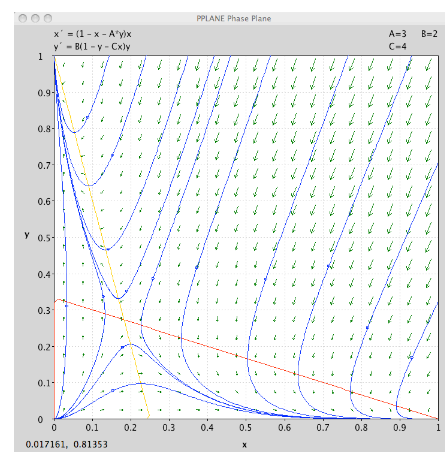




- Why does Species 1 dominate in the plot on the left, and Species 2 dominate in the plot on the right?
- Is it possible for both species to co-exist in perpetuity?
- How do birth/death rates and competition coefficients affect which species dominates?

Ex. #2: Lotka-Volterra comp.

- What happens if the system starts at x_0 = (1,0)?
- At x_0 = (0,1)?
- **Multiple equilibrium points** are possible in nonlinear systems



Ex. #2: Lotka-Volterra comp.

- To find all equilibria of the normalized Lotka-Volterra competition system, solve for all (x_1, x_2) such that

$$\begin{aligned} 0 &= x_1(1 - x_1 - Ax_2) \\ 0 &= x_2 B(1 - x_2 - Cx_1) \end{aligned}$$

- The 4 equilibria (for non-negative x_1, x_2) are

$$x^* = \begin{cases} (0,0) \\ (1,0) \\ (0,1) \\ \left(\frac{1-A}{1-AC}, \frac{1-C}{1-AC}\right) \end{cases}$$

Linearization about an Eq. Pt.

- Given the nonlinear system

$$\dot{x} = f(x), \; x \in \mathbb{R}^n, \; f : \mathbb{R}^n \to \mathbb{R}^n$$

with equilibrium point x*, consider a nearby point

$$x = \Delta x + x^*$$

- Use a Taylor's series 1st order approximation to determine the dynamics of (x-x*)

$$\begin{aligned} f(x) &= f(x^*) + \frac{\partial f}{\partial x}|_{x^*} \cdot \Delta x + \text{ H.O.T} \\ \dot{\Delta x} &\approx \frac{\partial f}{\partial x}|_{x^*} \cdot \Delta x \end{aligned}$$

- These dynamics MAY indicate what happens near the equilibrium point.

Linearization about an Eq. Pt.

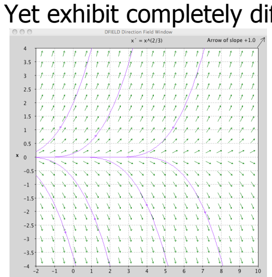
- As an example of when linearization fails, consider the following two systems:

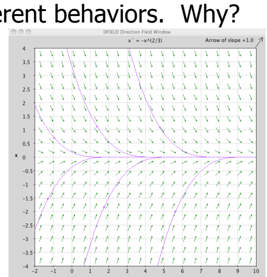
$$(a)\; \dot{x} = x^{2/3}, \qquad (b)\; \dot{x} = -x^{2/3}$$

- Which have the same linearization about 0

$$(a)\; \frac{\partial f}{\partial x}|_{x=0} = \frac{2}{3}x^{-1/3} = 0, \quad (b)\; \frac{\partial f}{\partial x}|_{x=0} = -\frac{2}{3}x^{-1/3} = 0$$

- Yet exhibit completely different behaviors. Why?





Linearization about an Eq. Pt.

- The eigenvalues of the Jacobian $D_f = \partial f/\partial x|_{x^*}$ provide information about the behavior of trajectories near the equilibrium point.
- Consider a 2D system

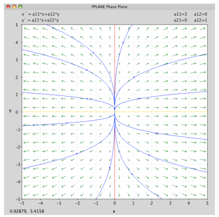
$$\Delta\dot{x} = D_f \Delta x$$

with two real, negative eigenvalues $\lambda_1, \lambda_2 < 0$

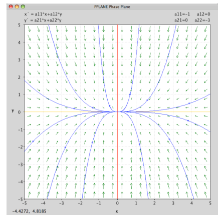
- Time can be 'eliminated' by examining Δx_1 vs. Δx_2 -- this is a **phase portrait** or a **phase-plane plot.**

Phase portraits of 2D linear systems

Stable node
$\lambda_1,\lambda_2<0$

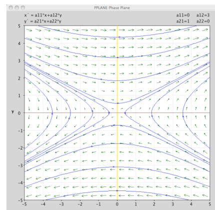


Unstable node
$\lambda_1,\lambda_2>0$



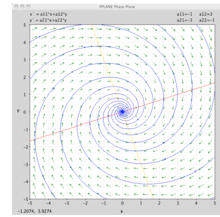
In all of these cases,
$\lambda_1, \lambda_2 \in \mathbb{R}$

Saddle
$\lambda_1>0$,
$\lambda_2<0$

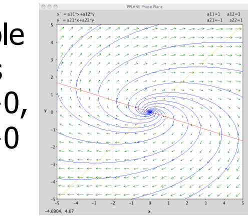


Phase portraits of 2D linear systems

Stable focus
$Re(\lambda_1)<0$,
$Re(\lambda_2)<0$

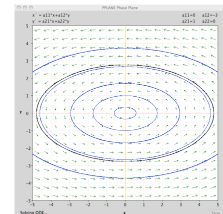


Unstable focus
$Re(\lambda_1)>0$,
$Re(\lambda_2)>0$



In all of these cases,
$\lambda_1, \lambda_2 \in \mathbb{C}$

Center
$Re(\lambda_1)=0$,
$Re(\lambda_2)=0$



Phase portraits of 2D linear systems

- Stable equilibrium points have trajectories that tend towards the origin and can be classified by eigenvalue characteristics:
  - Both negative real numbers (**stable node**)
  - Complex conjugate pair with negative real part (**stable focus**)

- Unstable equilibrium points have trajectories that tend towards infinity and can be classified by eigenvalue characteristics:
  - Both positive real numbers (**unstable node**)
  - Complex conjugate pair with positive real part (**unstable focus**)
  - Positive and negative real numbers (**saddle**)

- What happens when one eigenvalue has 0 real part?
- What happens when both eigenvalues have 0 real part?
- What happens when eigenvalues are repeated?

Big picture for this course

- When analyzing nonlinear systems, it is often useful to quantify
  - Stability -- e.g., steady-state behavior
  - Changes in stability -- e.g., due to variations in initial conditions or model parameters, model uncertainty, external disturbances, …
  - Invariance -- ability of the state to stay within a region of the state-space

- When controlling nonlinear systems, it is often useful to quantify
  - How the closed-loop system should behave
  - How much of the open-loop system's behavior can be changed
  - Sensitivity to errors, disturbances, computational time…
  - Actuation effort

- **The goal of this course is to develop a set of basic tools to analyze and control nonlinear systems**.