Lecture 2:
Second-order Systems

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Outline

- Techniques based on linear systems analysis
  - Phase-plane analysis
  - Example: Neanderthal / Early man competition
  - Hartman-Grobman theorem -- validity of linearizations
  - Example: Duffing's equation
- Nonlinear analysis of 2nd order systems
  - Closed orbits
  - Bendixon's theorem
  - Limit cycles
  - Index theory and Poincare theorem
  - Poincare-Bendixon theorem

Learning Objectives

- Stability through linearization
  - Compute the similarity transformation that results in a Jordan-form state matrix
  - Analyze a linear system in modal coordinates and relate to stability of nonlinear system near an equilibrium point (Hartmann-Grobman)
- Closed orbits and limit cycles in 2D
  - Distinguish between a closed orbit and a limit cycle
  - Determine whether a closed orbit exists (Bendixson's)
  - Determine whether a closed orbit exists in a given set of the state-space (Poincare-Bendixson)
  - Use Index theory to determine how many equilibria exist in a region of the state-space (n-dimensions)

References

- Khalil
  - Section 1.2
  - Section 2.1, 2.3-2.6
- Sastry
  - Section 2.2, 2.3

Stability through Linearization

- **Goal:** Determine stability of the nonlinear system around a given equilibrium point.
- Method: Analyze the linear system (an easy problem) and use Hartman-Grobman Theorem to make claims about the nonlinear system (a harder problem)

Nonlinear system
$\dot{x} = f(x)$

Linearization about an equilibrium point
$\Delta x = x - x^*$

Linear system
$\Delta \dot{x} = \frac{\partial f}{\partial x}|_{x=x^*} \cdot \Delta x$
(dynamics of deviations from the equilibrium point)

- The catch: Cannot apply blindly!

Stability through Linearization

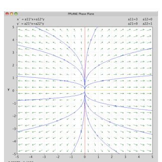
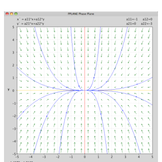
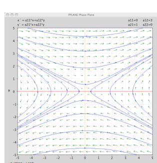
- Any linear system can be transformed via a **similarity transformation** into **real Jordan form**

$$\dot{x} = Ax \xrightarrow[T^{-1}AT = J]{z = T^{-1}x} \dot{z} = Jz$$

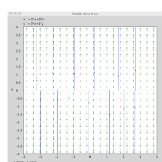
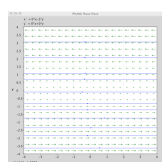
- The phase portrait of the system in modal coordinates depends on the form of J

Stability through Linearization

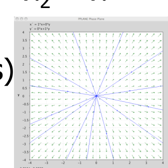
- Case 1a: $J = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$ $\lambda_1 \neq \lambda_2$

- Case 1b: $\lambda_1=0$ or $\lambda_2=0$

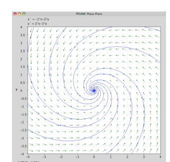
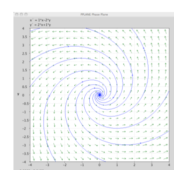
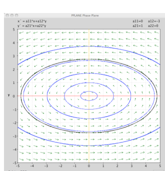



- Case 1c: $\lambda_1 = \lambda_2 = \lambda$ (independent eigenvectors)

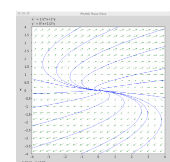
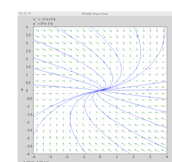


Stability through Linearization

- Case 2: $J = \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix}$

- Case 3: $J = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$ (dependent eigenvectors)

Ex. #3: Neanderthal extinction

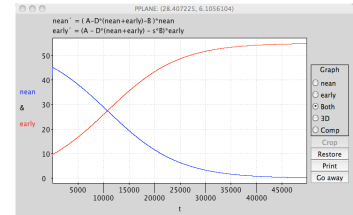
- Similar to Lotka-Volterra competition equations
- Neanderthals competed with Early Modern Man approximately 50,000 years ago

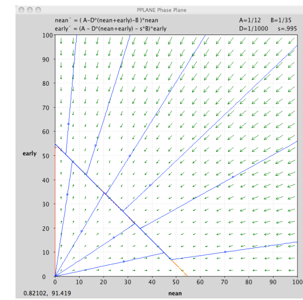
$$\begin{aligned}\dot{x}_N &= x_N(A - B - D(x_N + x_E)) \\ \dot{x}_E &= x_E(A - sB - D(x_N + x_E))\end{aligned}$$

- Parameters
  - Birth rate *A*
  - Death rate *B* (Neanderthals), *s·B* (Early Man)
  - Competition coefficient *D*
  - Similarity parameter *s*
- J. C. Flores, "A Mathematical Model for Neanderthal Extinction", Journal of Theoretical Biology, Volume 191, Issue 3, 7 April 1998, Pages 295-298.

Ex. #3: Neanderthal extinction

- Show that the equilibria are
  - (0,0),
  - (0,(A-sB)/D),
  - ((A-B)/D, 0)
- Compute the eigenvalues of the Jacobian matrix at each isolated equilibium point.
- Show that the only stable equilibrium point corresponds to Neanderthals' extinction and a surviving population of Early Modern Man.





Hartman-Grobman Theorem

If the linearization about equilibrium point *x**

$$\Delta\dot{x} = \frac{\partial f}{\partial x}\Big|_{x^*}\Delta x \qquad (1)$$

of the nonlinear system

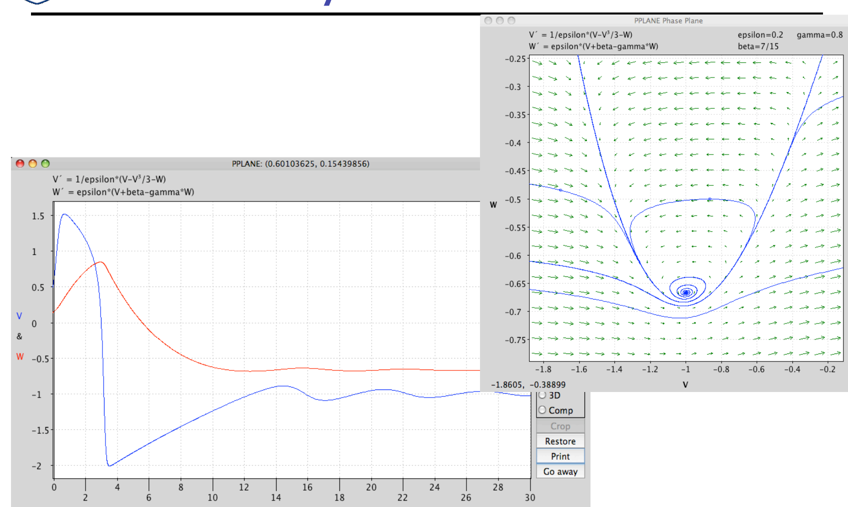
$$\dot{x} = f(x),\ x \in \mathbb{R}^n \qquad (2)$$

Has no eigenvalues on the imaginary axis, then there exists a continuous map h (that has a continuous inverse),

$$h:\ B(x\ ,\delta) \rightarrow \mathbb{R}^n$$

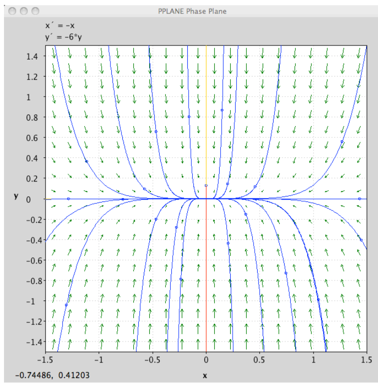
Which takes trajectories from (2) and maps them onto those of (1).

Heart Rhythms: stable foci

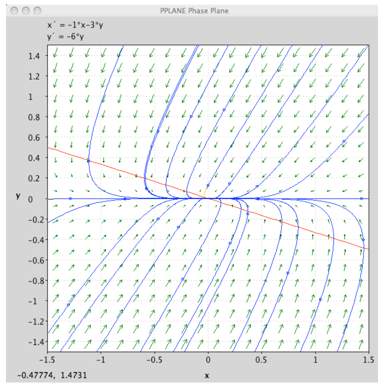


Lotka-Volterra Competition

(z_1, z_2)

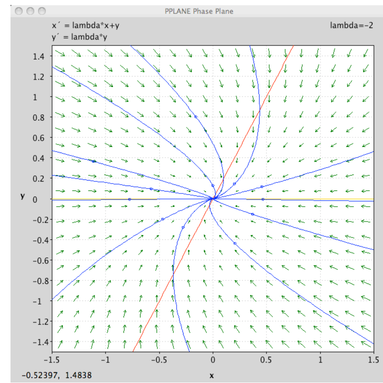


(ΔV,ΔW around (1,0))

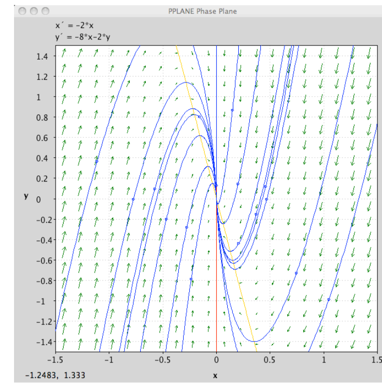


Lotka Volterra Competition

(z_1, z_2)

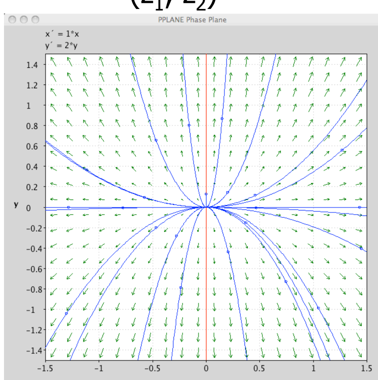


(ΔV,ΔW around (0,1))

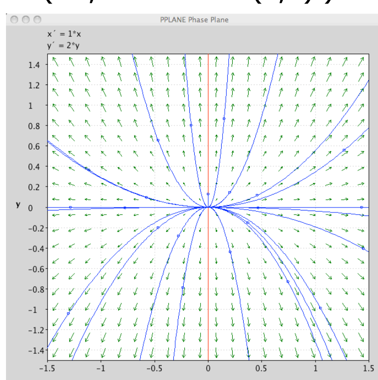


Lotka-Volterra Competition

(z_1, z_2)

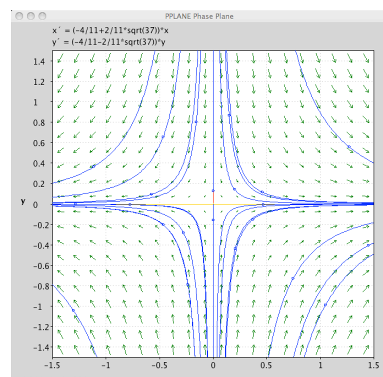


(ΔV,ΔW around (0,0))

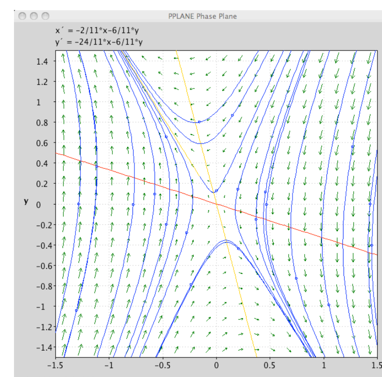


Lotka-Volterra Competition

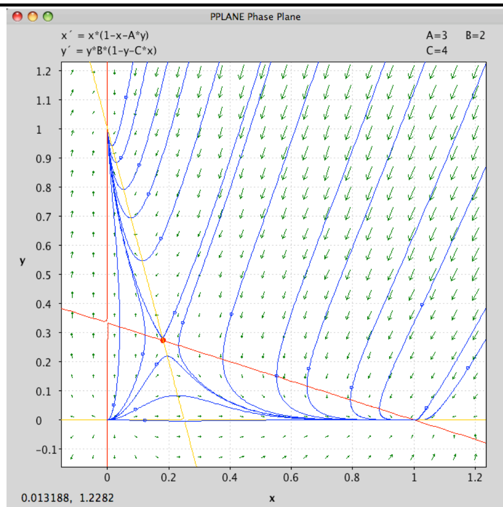
(z_1, z_2)



(ΔV,ΔW around (2/11,3/11))

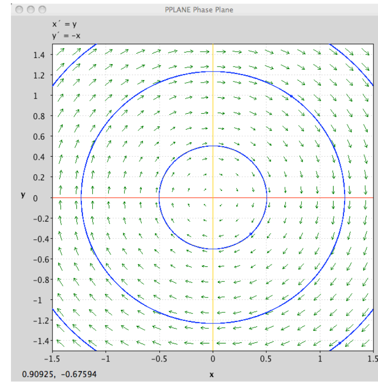


Lotka-Volterra Competition

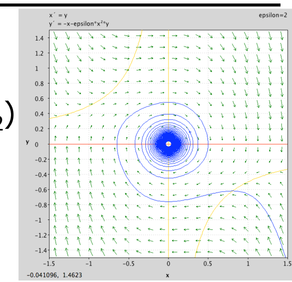


Non-hyperbolic eq. pt

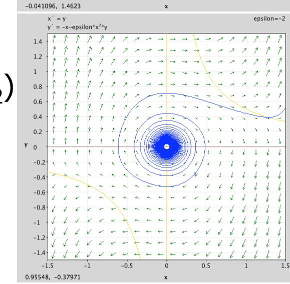
(z_1, z_2)



(x_1, x_2)
$\varepsilon>0$



(x_1, x_2)
$\varepsilon<0$



Phase-plane analysis

- Process for each equilibrium point:
  - Linearize nonlinear system about equilibrium point.
  - Compute eigenvalues and eigenvectors of D_f.
  - Create similarity transformation.
  - Sketch phase-plane plot in transformed coordinates.
  - If eigenvalues do NOT have 0 real part, stability of linear system is same as stability of nonlinear system near the equilibrium point.
- Note that pplane5 plots are **simulations**. They do not provide any guarantees of the actual system's behavior.

Closed orbits of planar systems

- Closed orbits vs. limit cycles
- Bendixon's theorem
  - Absence of closed orbits
- Poincare-Bendixson theorem
  - Existence of closed orbits within an invariant set
- Index theory
- Poincare theorem
  - # of equilibria within a closed orbit

Closed orbits of planar systems

Definition: Closed orbit

- A closed orbit γ of the planar system

$$\dot{x} = f(x),\ x \in \mathbb{R}^2$$

is the trace of the trajectory of a non-trivial periodic solution.

Notes:

- An equilibrium point cannot be a closed orbit
- For any point $x_0 \in \gamma$, there exists a finite value *T* such that $x(nT) = x_0,\ n \in \mathbb{Z}$

Closed orbits of planar systems

Definition: Limit cycle

- A limit cycle is a closed orbit g such that there is a $x(t) \notin \gamma$ such that

$$x(t) \to \gamma \text{ as } t \to \infty \text{ or } t \to -\infty$$

(stable) (unstable)

Note:

- All limit cycles are closed orbits, but not all closed orbits are limit cycles.
- The theorems to follow concern closed orbits, not limit cycles necessarily.

Closed orbits of planar systems

- Bendixson's theorem

Suppose *D* is a simply connected region in R^2, such that

$$\mathrm{div}(f) \triangleq \frac{\partial f}{\partial x_1} + \frac{\partial f}{\partial x_2} \neq 0$$

and div(f) does not change sign in D.

Then *D* contains no closed orbits of

$$\dot{x} = f(x),\ x \in \mathbb{R}^2$$

Notes:

- "Simply connected" means one region with no holes.
- Proves the absence of closed orbits

Closed orbits of planar systems

- Poincare-Bendixson Theorem

Consider the planar dynamical system

$$\dot{x} = f(x),\ x \in \mathbb{R}^2$$

Every closed, bounded, non-empty, positively invariant set $K \subseteq \mathbb{R}^2$ contains an equilibrium point or a closed orbit.

Notes:

- If *K* contains equilibrium points, *K* may also contain a union of trajectories connecting these equilibrium points, but this is not create a closed orbit.
- An open set *U* that is enclosed by a closed orbit contains an eq. point and possibly a closed orbit.

Index theory

- Definition: The index of a region *D* with respect to *f(x)* is defined as

$$\begin{aligned} I_f(D) &= \tfrac{1}{2\pi}\int_J d\theta_f(x) \\ &= \tfrac{1}{2\pi}\int_J \tfrac{f_1 df_2 - f_2 df_1}{f_1^2+f_2^2} \end{aligned}$$

  with

$$\theta_f(x) = \tan^{-1}\frac{f_2}{f_1}$$

- Notes:
  - θ_f is the angle made by *f(x)* with the x_1 axis.
  - $I_f(D)$ is an integer
  - For *D* that contains a single equilibrium point x_0, we refer to the index of the equilibrium point at $I_f(x_0)$

Index theory

- Poincare theorem:

  Let *N* represent the number of nodes, centers, and foci enclosed by a closed orbit, and let *S* represent the number of enclosed saddle points. Then

$$N = S + 1$$

- Notes:
  - Index theory predicts the existence of equilibrium points without doing detailed calculations.