EECE 574 - Adaptive Control

Model-Reference Adaptive Control - Part I

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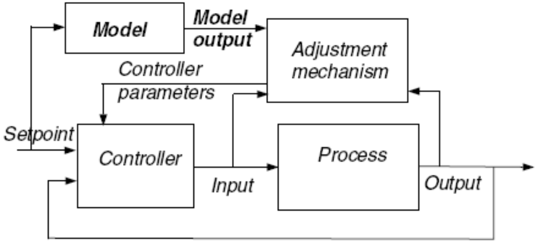
Model-Reference Adaptive Systems

The MRAC or MRAS is an important adaptive control methodology [1]



[1] see Chapter 5 of the Åström and Wittenmark textbook, or H. Butler, "Model-Reference Adaptive Control-From Theory to Practice", Prentice-Hall, 1992

Model-Reference Adaptive Systems

- The MIT rule
- Lyapunov stability theory
- Design of MRAS based on Lyapunov stability theory
- Hyperstability and passivity theory
- The error model
- Augmented error
- A model-following MRAS

The MIT Rule

- Original approach to MRAC developed around 1960 at MIT for aerospace applications
- With $e = y - y_m$, adjust the parameters θ to minimize

$$J(\theta) = \frac{1}{2}e^2$$

- It is reasonable to adjust the parameters in the direction of the negative gradient of J:

$$\frac{d\theta}{dt} = -\gamma\frac{\partial J}{\partial \theta} = -\gamma e\frac{\partial e}{\partial \theta}$$

- $\partial e/\partial \theta$ is called the sensitivity derivative of the system and is evaluated under the assumption that θ varies slowly

The MIT Rule

- The derivative of J is then described by

$$\frac{dJ}{dt} = e\frac{\partial e}{\partial t} = -\gamma e^2\left(\frac{\partial e}{\partial \theta}\right)^2$$

- Alternatively, one may consider $J(e) = |e|$ in which case

$$\frac{d\theta}{dt} = -\gamma\frac{\partial J}{\partial \theta} = -\gamma\frac{\partial e}{\partial \theta}\mathrm{sign(e)}$$

- The sign-sign algorithm used in telecommunications where simple implementation and fast computations are required, is

$$\frac{d\theta}{dt} = -\gamma\frac{\partial J}{\partial \theta} = -\gamma\mathrm{sign}\left(\frac{\partial \mathrm{e}}{\partial \theta}\right)\mathrm{sign(e)}$$

MIT Rule: Example 1

- Process: $y = kG(s)$ where $G(s)$ is known but k is unknown
- The desired response is $y_m = k_0 G(s) u_c$
- Controller is $u = \theta u_c$
- Then $e = y - y_m = kG(p)\theta u_c - k_0 G(p) u_c$
- Sensitivity derivative

$$\frac{\partial e}{\partial \theta} = kG(p)u_c = \frac{k}{k_0} y_m$$

- MIT rule

$$\frac{d\theta}{dt} = \gamma' \frac{k}{k_0} y_m e = -\gamma y_m e$$

MIT Rule: Example 1

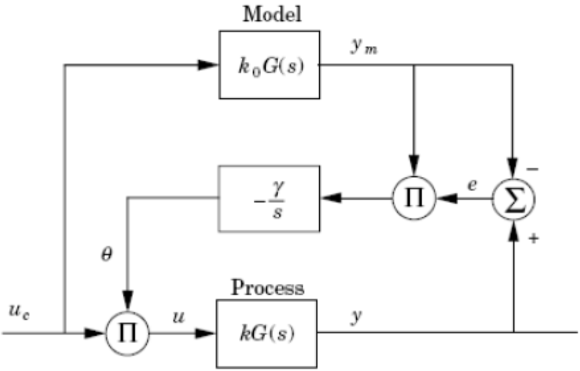


Figure: MIT rule for adjustment of feedforward gain (from textbook).

MIT Rule: Example 1

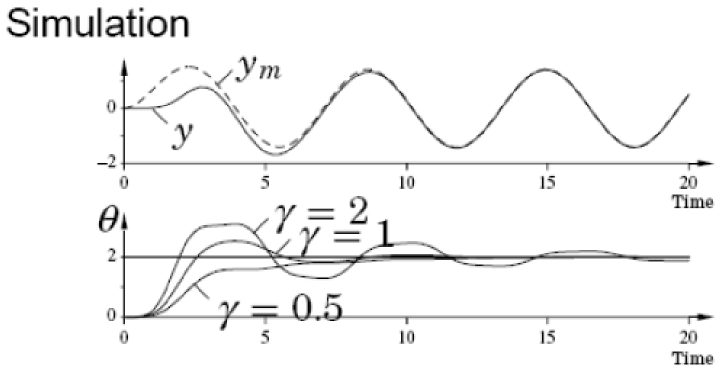


Figure: MIT rule for adjustment of feedforward gain: Simulation results (from textbook).

MIT Rule: Example 2

Consider the first-order system

$$\frac{dy}{dt} = -ay + bu$$

The desired closed-loop system is

$$\frac{dy_m}{dt} = -a_m y_m + b_m u_c$$

Applying model-following design (see lecture notes on pole placement)

$$\deg A_0 \geq 0 \qquad \deg S = \deg R = \deg T = 0$$

MIT Rule: Example 2

The controller is then

$$u(t) = t_0 u_c(t) - s_0 y(t)$$

For perfect model-following

$$\begin{aligned}\frac{dy}{dt} &= -ay(t) + b[t_0 u_c(t) - s_0 y(t)] \\ &= -(a + bs_0)y(t) + bt_0 u_c \\ &= -a_m y_m(t) + b_m u_c\end{aligned}$$

This implies

$$\begin{aligned}s_0 &= \frac{a_m - a}{b} \\ t_0 &= \frac{b_m}{b}\end{aligned}$$

MIT Rule: Example 2

With the controller we can write[2]

$$y = \frac{bt_0}{p+a+bs_0} u_c$$

With $e = y - y_m$, the sensitivity derivatives are

$$\begin{aligned}
\frac{\partial e}{\partial t_0} &= \frac{b}{p+a+bs_0} u_c \\
\frac{\partial e}{\partial s_0} &= \frac{b^2 t_0}{(p+a+bs_0)^2} u_c = \frac{b}{p+a+bs_0} y
\end{aligned}$$

[2] p is the differential operator $d(.)/dt$

MIT Rule: Example 2

However, a and b are unknown[3]. For the nominal parameters, we know that

$$a + bs_0 = a_m$$

$$p + a + bs_0 \approx p + a_m$$

Then

$$\begin{aligned} \frac{dt_0}{dt} &= -\gamma' b \left(\frac{1}{p + a_m} u_c \right) e = -\gamma \left(\frac{a_m}{p + a_m} u_c \right) e \\ \frac{ds_0}{dt} &= \gamma' b \left(\frac{1}{p + a_m} y \right) e = \gamma \left(\frac{a_m}{p + a_m} y \right) e \end{aligned}$$

with $\gamma = \gamma' b / a_m$, i.e. b is absorbed in γ and the filter is normalized with static gain of one.

[3] after all, we want to design an adaptive controller!

MIT Rule: Example 2

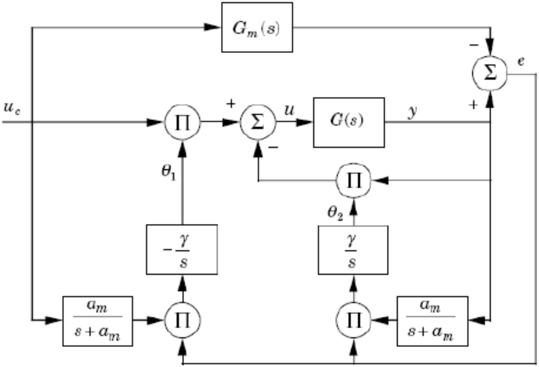


Figure: MIT rule for first-order (from textbook)

MIT Rule: Example 2

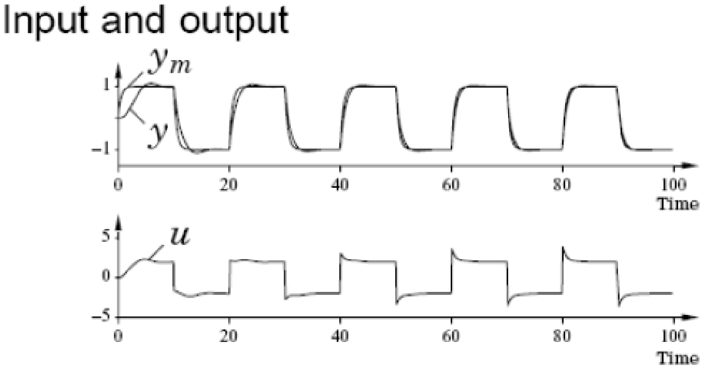


Figure: Simulation of MIT rule for first-order with $a = 1$, $b = 0.5$, $a_m = b_m = 2$ and $\gamma = 1$(from textbook)

MIT Rule: Example 2

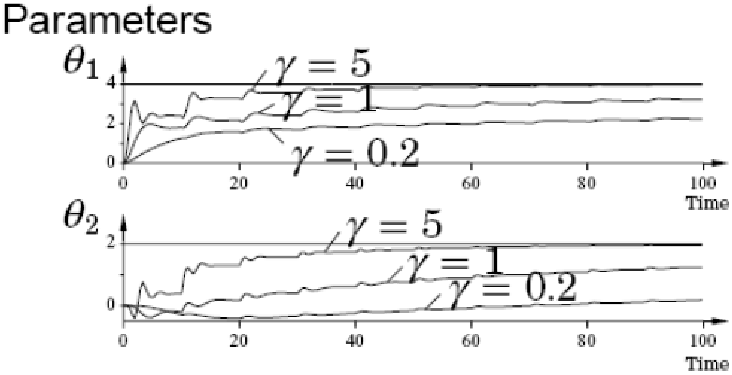


Figure: Simulation of MIT rule for first-order with $a = 1$, $b = 0.5$, $a_m = b_m = 2$. Controller parameters for $\gamma = 0.2$, 1 and 5(from textbook)

MIT Rule: Example 2

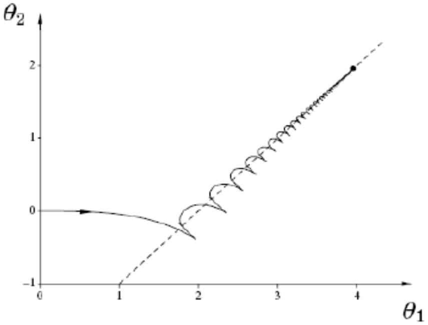


Figure: Simulation of MIT rule for first-order with $a = 1$, $b = 0.5$, $a_m = b_m = 2$. Relation between controller parameters θ_1 and θ_2. The dashed line is $\theta_2 = \theta_1 - a/b$. (from textbook)

Stability of the Adaptive Loop

Consider again the adaptation of a feedforward gain

$$G = \frac{1}{s^2 + a_1 s + a_2}$$

$$e = (\theta - \theta_0) G u_c$$

$$\begin{aligned} \frac{\partial e}{\partial \theta} &= G u_c = \frac{y_m}{\theta_0} \\ \frac{d\theta}{dt} &= -\gamma' e \frac{\partial e}{\partial \theta} = -\gamma' e \frac{y_m}{\theta_0} = -\gamma e y_m \end{aligned}$$

Stability of the Adaptive Loop

Thus the system can be described as

$$\begin{aligned}
\frac{d^2 y_m}{dt^2} + a_1 \frac{dy_m}{dt} + a_2 y_m &= \theta_0 u_c \\
\frac{d^2 y}{dt^2} + a_1 \frac{dy}{dt} + a_2 y &= \theta u_c \\
\frac{d\theta}{dt} &= -\gamma (y - y_m) y_m
\end{aligned}$$

This is difficult to solve analytically

Stability of the Adaptive Loop

$$\frac{d^3y}{dt^3} + a_1\frac{d^2y}{dt^2} + a_2\frac{dy}{dt} + \gamma u_c y_m y = \theta\frac{du_c}{dt} + \gamma u_c y_m^2$$

Assume u_c^o and y_m^o constant, equilibrium is

$$y(t) = y_m^o = \frac{\theta_0 u_c^o}{a_2}$$

which is stable if

$$a_1 a_2 > \gamma u_c^o y_m^o = \frac{\gamma}{a_2}(u_c^o)^2$$

Thus, if γ or u_c is sufficiently large, the system will be unstable

Stability of the Adaptive Loop

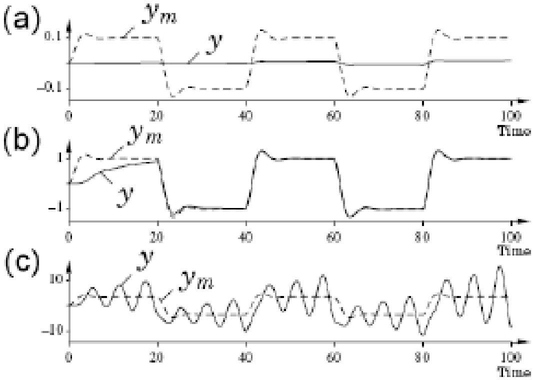


Figure: Influence on convergence and stability of signal amplitude for MIT rule (from textbook)

Modified MIT Rule

- Normalization can be used to protect against dependence on the signal amplitudes
- With $\varphi = \partial e/\partial\theta$, the MIT rule can be written as

$$\frac{d\theta}{dt} = -\gamma\varphi e$$

- The **normalized** MIT rule is then

$$\frac{d\theta}{dt} = \frac{-\gamma\varphi e}{\alpha^2 + \varphi^T\varphi}$$

- For the previous example,

$$\frac{d\theta}{dt} = \frac{-\gamma e y_m/\theta_0}{\alpha^2 + y_m^2/\theta_0^2}$$

Modified MIT Rule

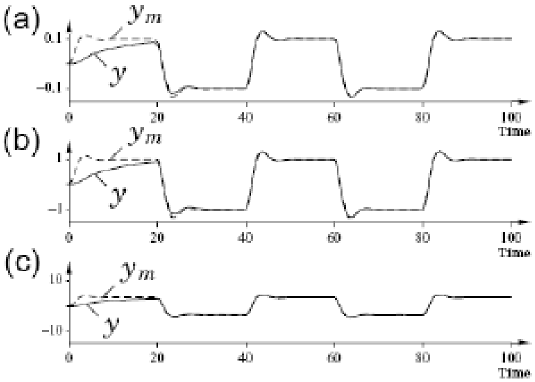


Figure: Influence on convergence and stability of signal amplitude for modified MIT rule (from textbook)

Lyapunov Stability Theory

- Aleksandr M Lyapunov (1857-1918, Russia)
- Classmate of Markov, taught by Chebyshev
- Doctoral thesis *The general problem of the stability of motion* became a fundamental contribution to the study of dynamic systems stability
- He shot himself three days after his wife died of tuberculosis

Lyapunov Stability Theory

- Consider the nonlinear time-varying system

$$\dot{x} = f(x,t) \qquad \text{with } f(0,t) = 0$$

- If at time $t = t_0$ $\|x(0)\| = \delta$, i.e. if the initial state is not the equlibrium state $x^* = 0$, what will happen?
- There are four possibilities

Lyapunov Stability Theory

- The system is stable. For a sufficiently small δ, x stays within ε of x^*. If δ can be chosen independently of t_0, then the system is said to be uniformly stable.
- The system is unstable. It is possible to find ε which does not allow any δ.
- The system is asymptotically stable. For $\delta < R$ and an arbitrary ε, there exists t^* such that for all $t > t^*$, $\|x - x^*\| < \varepsilon$. It implies that $\|x - x^*\| \longrightarrow 0$ as $t \longrightarrow \infty$.
- If asymptotic stability is guaranteed for any δ, the system is globally asymptotically stable.

Lyapunov Stability Theory

- A scalar time-varying function is locally positive definite if $V(o,t) = 0$ and there exists a time-invariant positive definite function $V_0(x)$ such that $\forall t \geq t_0$, $V(x,t) \geq V_0(x)$. In other words, it dominates a time-invariant positive definite function.
- $V(x,t)$ is radially unbounded if $0 < \alpha\|x\| \leq V(x,t)$, $\alpha > 0$.
- A scalar time-varying function is said to be decrescent if $V(o,t) = 0$ and there exists a time-invariant positive definite function $V_1(x)$ such that $\forall t \geq t_0$, $V(x,t) \leq V_1(x)$.
- In other words, it is dominated by a time-invariant positive definite function.

Lyapunov Stability Theory

Theorem (Lyapunov Theorem)

- ***Stability:*** *if in a ball B_R around the equilibrium point* 0*, there exists a scalar function $V(x,t)$ with continuous partial derivatives such that*

  1. *V is positive definite*
  2. *$\dot{V}$ is negative semi-definite*

  *then the equilibrium point is **stable**.*

- ***Uniform stability and uniform asymptotic stability:*** *If furthermore,*

  3. *V is decrescent*

  *then the origin is **uniformly stable**. If condition 2 is strengthened by requiring that $\dot{V}$ be negative definite, then the equilibrium is **uniformly asymptotically stable**.*

- ***Global uniform asymptotic stability:*** *If B_R is replaced by the whole state space, and condition 1, the strengthened condition 2, condition 3 and the condition*

  4. *V(x,t) is radially unbounded*

  are all satisfied, then the equilibrium point at 0 *is **globally uniformly asymptotically stable**.*

Lyapunov Stability Theory

- The Lyapunov function has similarities with the **energy content** of the system and must be **decreasing** with time.
- This result can be used in the following way to design a stable a stable adaptive controller
  1. First, the **error equation**, i.e. a differential equation describing either the output error $y - y_m$ or the state error $x - x_m$ is derived.
  2. Second, a Lyapunov function, function of both the signal error $e = x - x_m$ and the parameter error $\phi = \theta - \theta_m$ is chosen. A typical choice is

     $$V = e^T P e + \phi^T \Gamma^{-1} \phi$$

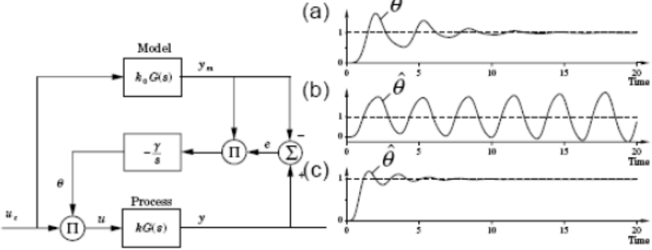
     where both P and Γ^{-1} are positive definite matrices.
  3. The time derivative of V is calculated. Typically, it will have the form

     $$\dot{V} = -e^T Q e + \text{ some terms including } \phi$$

     Putting the extra terms to zero will ensure negative definiteness for $\dot{V}$ if Q is positive definite, and will provide the adaptive law.

Stability of Adaptive Loop

Consider the MRAS designed with MIT rule



$u_c = sin\omega t$ $\omega =$ 1(a), 2(b) and 3(c)

Lyapunov Design of MRAC

- Determine controller structure
- Derive the error equation
- Find a Lyapunov equation
- Determine adaptation law that satisfies Lyapunov theorem

Adaptation of Feedforward Gain

- Process model

$$\frac{dy}{dt} = -ay + ku$$

- Desired response

$$\frac{dy_m}{dt} = -ay_m + k_0 u_c$$

- Controller

$$u = \theta u_c$$

- Introduce error $e = y - y_m$
- The error equation is

$$\frac{de}{dt} = -ae + (k\theta - k_0)u_c$$

Adaptation of Feedforward Gain

- System model

$$\begin{aligned} \frac{dy}{dt} &= -ay + ku \\ \frac{d\theta}{dt} &= ?? \end{aligned}$$

- Desired equilibrium

$$\begin{aligned} e &= 0 \\ \theta &= \theta_0 = \frac{k_0}{k} \end{aligned}$$

Adaptation of Feedforward Gain

- Consider the Lyapunov function

$$\begin{aligned} V(e,\theta) &= \frac{\gamma}{2}e^2 + \frac{k}{2}(\theta - \theta_0)^2 \\ \frac{dV}{dt} &= -\gamma a e^2 + (k\theta - k_0)(\frac{d\theta}{dt} + \gamma u_c e) \end{aligned}$$

- Choosing the adjustment rule

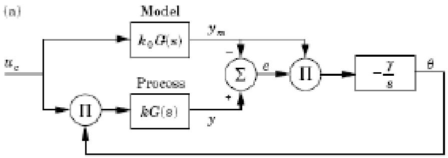
$$\frac{d\theta}{dt} = -\gamma u_c e$$

gives

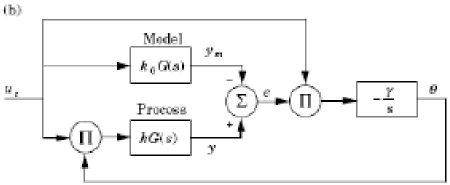
$$\frac{dV}{dt} = -\gamma a e^2$$

Adaptation of Feedforward Gain

Lyapunov rule: $\frac{d\theta}{dt} = -\gamma u_c e$

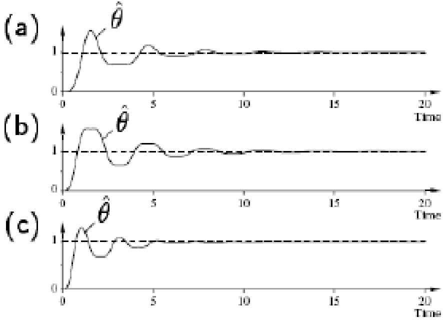


MIT rule: $\frac{d\theta}{dt} = -\gamma y_m e$

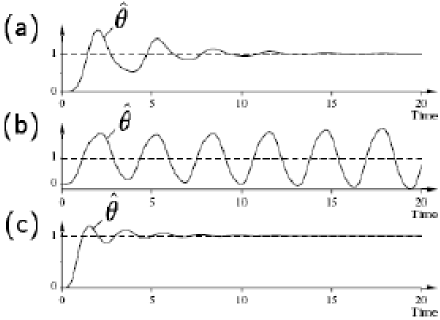


Adaptation of Feedforward Gain

Lyapunov rule:



MIT rule:



First-order System

- Process model

$$\frac{dy}{dt} = -ay + bu$$

- Desired response

$$\frac{dy_m}{dt} = -a_m y_m + b_m u_c$$

- Controller

$$u = \theta_1 u_c - \theta_2 y$$

- Introduce error $e = y - y_m$
- The error equation is

$$\frac{de}{dt} = -a_m e - (b\theta_2 + a - a_m)y + (b\theta_1 - b_m)u_c$$

First-order System

- Candidate Lyapunov function

$$V(t,\theta_1,\theta_2) = \frac{1}{2}\left(e^2 + \frac{1}{b\gamma}(b\theta_2 + a - a_m)^2 + \frac{1}{b\gamma}(b\theta_1 - b_m)^2\right)$$

- Derivative

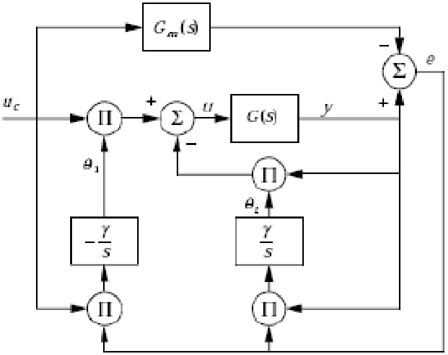
$$\begin{aligned}\frac{dV}{dt} &= \frac{de}{dt} + \frac{1}{\gamma}(b\theta_2 + a - a_m)\frac{d\theta_2}{dt} + \frac{1}{\gamma}(b\theta_1 - b_m)\frac{d\theta_1}{dt} \\ &= -a_m e^2 + \frac{1}{\gamma}(b\theta_2 + a - a_m)(\frac{d\theta_2}{dt} - \gamma y e) + \frac{1}{\gamma}(b\theta_1 - b_m)(\frac{d\theta_1}{dt} + \gamma u_c e)\end{aligned}$$

- This suggests the adaptation law

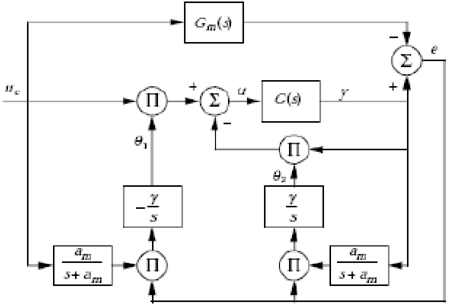
$$\begin{aligned}\frac{d\theta_1}{dt} &= -\gamma u_c e \\ \frac{d\theta_2}{dt} &= \gamma y e\end{aligned}$$

First-order System

Lyapunov Rule

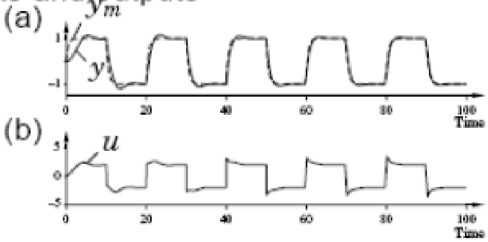


MIT Rule

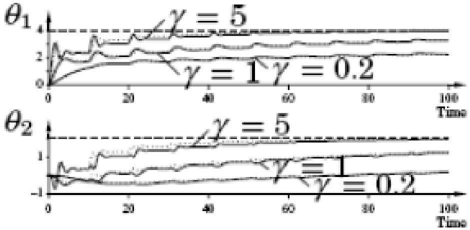


First-order System

Process inputs and outputs



Parameters



Lyapunov-based MRAC Design

- The main advantage of Lyapunov design is that it guarantees a closed-loop system.
- For a linear, asymptotically stable governed by a matrix A, a positive symmetric matrix Q yields a positive symmetric matrix P by the equation

$$A^T P + PA = -Q$$

This equation is known as Lyapunov's equation.

Lyapunov-based MRAC Design

- The main drawback of Lyapunov design is that there is no systematic way of finding a suitable Lyapunov function V leading to a specific adaptive law.
- For example, if one wants to add a proportional term to the adaptive law, it is not trivial to find the corresponding Lyapunov function.
- The hyperstability approach is more flexible than the Lyapunov approach.