THE UNIVERSITY OF BRITISH COLUMBIA

Public Key Cryptography

EECE 412

What is it?

- Two keys
  - Sender uses recipient's **public key** to encrypt
  - Receiver uses his **private key** to decrypt
- Based on **trap door, one way function**
  - Easy to compute in one direction
  - Hard to compute in other direction
  - "Trap door" used to create keys
  - Example: Given p and q, product N=pq is easy to compute, but given N, it is hard to find p and q

How is it used?

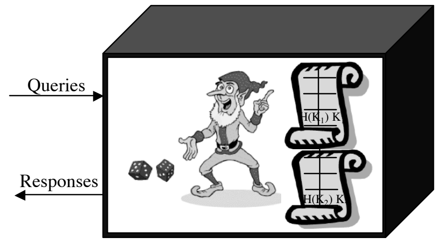
- Encryption
  - Suppose we encrypt M with Bob’s public key
  - Only Bob’s private key can decrypt to find M
- Digital Signature
  - **Sign** by “encrypting” with private key
  - Anyone can **verify** signature by “decrypting” with public key
  - But only private key holder could have signed
  - Like a handwritten signature (and then some)

Topic Outline

- The Random Oracle model for Public Key Cryptosystems
  - Public key encryption and trapdoor one-way permutations
  - Digital signatures
- Looking under the hood
  - Knapsack
  - RSA
- Uses of Public Crypto
- The order of sign and encrypt

Public Key Encryption and Trap-door One-Way Permutation
as Random Oracle

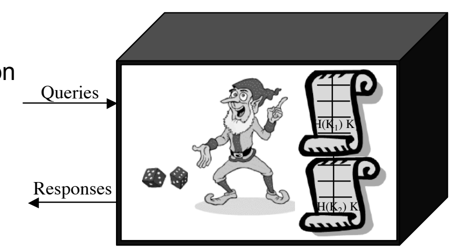
- Public Key Encryption Scheme:
  - Key pair (KR, KR^{-1}) generation function from random string R
    - KR → KR^{-1} is infeasible
  - C = $\{M\}_{KR}$
  - M = $\{C\}_{KR^{-1}}$



- In:
  - fixed size short string (plaintext) M,
  - Key KR
- Out: fixed size short string (ciphertext) C

Digital Signature as Random Oracle

- Public Key Signature Scheme:
  - Key pair (σR, VR) generation function
    - VR → σR is infeasible
  - S = $Sig_{\sigma R}(M)$
  - {True, False} = $Ver_{VR}(S)$



	Signing	Verifying
Input	Any string M + σR	S + VR
Output	S = hash(M) | cipher block	"True" or "False"

Looking Under the Hood

Knapsack

Knapsack Problem

- Given a set of n weights $W_0, W_1, \ldots, W_{n-1}$ and a sum S, is it possible to find $a_i \in \{0,1\}$ so that
  $S = a_0W_0 + a_1W_1 + \ldots + a_{n-1}W_{n-1}$
  (technically, this is "subset sum" problem)
- **Example**
  - Weights (62,93,26,52,166,48,91,141)
  - Problem: Find subset that sums to S=302
  - Answer: 62+26+166+48=302
- The (general) knapsack is NP-complete

Knapsack Problem

- General knapsack (GK) is hard to solve
- But **superincreasing knapsack** (SIK) is easy
- SIK each weight greater than the sum of all previous weights
- **Example**
  - Weights (2,3,7,14,30,57,120,251)
  - Problem: Find subset that sums to S=186
  - Work from largest to smallest weight
  - Answer: 120+57+7+2=186

Knapsack Cryptosystem

1. Generate superincreasing knapsack (SIK)
2. Convert SIK into "general" knapsack (GK)
3. **Public Key:** GK
4. **Private Key:** SIK plus conversion factors

- Easy to encrypt with GK
- With private key, easy to decrypt (convert ciphertext to SIK)
- Without private key, must solve GK (???)

Knapsack Cryptosystem

- Let (2,3,7,14,30,57,120,251) be the SIK
- Choose m = 41 and n = 491 with *m*, *n* relatively prime and *n* greater than sum of elements of SIK
- General knapsack
  - $2 \cdot 41 \bmod 491 = 82$
  - $3 \cdot 41 \bmod 491 = 123$
  - $7 \cdot 41 \bmod 491 = 287$
  - $14 \cdot 41 \bmod 491 = 83$
  - $30 \cdot 41 \bmod 491 = 248$
  - $57 \cdot 41 \bmod 491 = 373$
  - $120 \cdot 41 \bmod 491 = 10$
  - $251 \cdot 41 \bmod 491 = 471$
- General knapsack: (82,123,287,83,248,373,10,471)

Knapsack Example

- **Private key:** (2,3,7,14,30,57,120,251), n = 491, m^{-1}=12
  - m^{-1} mod n = 41^{-1} mod 491 = 12
- **Public key:** (82,123,287,83,248,373,10,471)
- **Throw away:** m = 41
- Example: Encrypt 10010110

  82 + 83 + 373 + 10 = 548
- To decrypt,
  - 548 · 12 = 193 mod 491
  - Solve (easy) SIK with S = 193
  - Obtain plaintext 10010110

Knapsack Weakness

- **Trapdoor:** Convert SIK into "general" knapsack using modular arithmetic
- **One-way:** General knapsack easy to encrypt, hard to solve; SIK easy to solve
- This knapsack cryptosystem is **insecure**
  - Broken in 1983 with Apple II computer
  - The attack uses **lattice reduction**
- "General knapsack" is not general enough!
- This special knapsack is easy to solve!

RSA

RSA

- Invented by Cocks (GCHQ), independently, by Rivest, Shamir and Adleman (MIT)
- Let p and q be two large prime numbers
- Let N = pq be the **modulus**
- Choose e relatively prime to (p-1)(q-1)
  - Totient: ϕ(N) = (p-1)(q-1), the number of numbers less than N that are relatively prime to N.
- Find d s.t. e*d = 1 mod (p-1)(q-1)
- **Public key:** is (N,e),
- **Private key:** is d
- **Throw Away:** p,q

RSA

- To encrypt message M compute
  - $C = M^e \bmod N$
- To decrypt C compute
  - $M = C^d \bmod N$
- Recall that e and N are public
- If attacker can factor N, he can use e to easily find d since *ed = 1 mod (p−1)(q−1)*
- Factoring the modulus breaks RSA
- It is not known whether factoring is the only way to break RSA

Simple RSA Example

- Example of RSA
  - Select "large" primes p = 11, q = 3
  - Then N = pq = 33 and (p−1)(q−1) = 20
  - Choose e = 3 (relatively prime to 20)
  - Find d such that ed = 1 mod 20, we find that d = 7 works
- **Public key:** (N, e) = (33, 3)
- **Private key:** d = 7

Simple RSA Example

- **Public key:** (N, e) = (33, 3)
- **Private key:** d = 7
- Suppose message M = 8
- Ciphertext C is computed as
  $C = M^e \bmod N = 8^3 = 512 = 17 \bmod 33$
- Decrypt C to recover the message M by
  $M = C^d \bmod N = 17^7 = 410{,}338{,}673$
  $= 12{,}434{,}505 * 33 + 8 = 8 \bmod 33$

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Uses for Public Key Crypto

Uses for Public Key Crypto

- Confidentiality
  - Transmitting data over insecure channel
  - Secure storage on insecure media
- Authentication
- Digital signature provides integrity and **non-repudiation**
  - No non-repudiation with symmetric keys

Non-non-repudiation

- Alice orders 100 shares of stock from Bob
- Alice computes **MAC** using symmetric key
- Stock drops, Alice claims she did not order
- Can Bob prove that Alice placed the order?
- **No!** Since Bob also knows symmetric key, he could have forged message
- **Problem:** Bob knows Alice placed the order, but he can't prove it

Non-repudiation

- Alice orders 100 shares of stock from Bob
- Alice **signs** order with her private key
- Stock drops, Alice claims she did not order
- Can Bob prove that Alice placed the order?
- **Yes!** Only someone with Alice's private key could have signed the order
- This assumes Alice's private key is not stolen (revocation problem)

Sign and Encrypt vs Encrypt and Sign

Public Key Notation

- **Sign** message M with Alice's **private key:** $[M]_{Alice}$
- **Encrypt** message M with Alice's **public key:** $\{M\}_{Alice}$
- Then

$\{[M]_{Alice}\}_{Alice} = M$

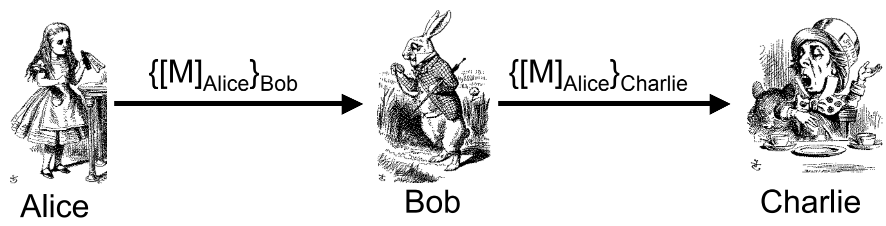
$[\{M\}_{Alice}]_{Alice} = M$

Confidentiality and Non-repudiation

- Suppose that we want confidentiality and non-repudiation
- Can public key crypto achieve both?
- Alice sends message to Bob
  - **Sign and encrypt** $\{[M]_{Alice}\}_{Bob}$
  - **Encrypt and sign** $[\{M\}_{Bob}]_{Alice}$
- Can the order possibly matter? (pg. 77-79 Stamp)

Sign and Encrypt

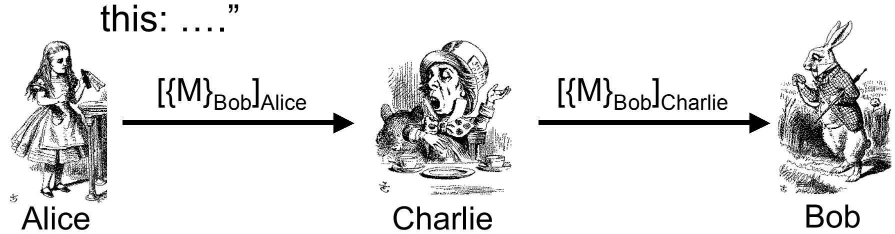
- M = “I love you”



- **Q:** What is the problem?
- **A:** Charlie misunderstands crypto!

Encrypt and Sign

- M = “My theory, which is mine, is this: ….”



- **Note** that Charlie cannot decrypt M
- **Q:** What is the problem?
- **A:** Bob misunderstands crypto!

Summary

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  - Digital signatures
- Looking under the hood
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- Uses of Public Crypto
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