Public Key Cryptography

EECE 412

What is it?

- Two keys
  - Sender uses recipient's **public key** to encrypt
  - Receiver uses his **private key** to decrypt
- Based on **trap door, one way function**
  - Easy to compute in one direction
  - Hard to compute in other direction
  - "Trap door" used to create keys
  - Example: Given p and q, product $N=pq$ is easy to compute, but given N, it is hard to find p and q

How is it used?

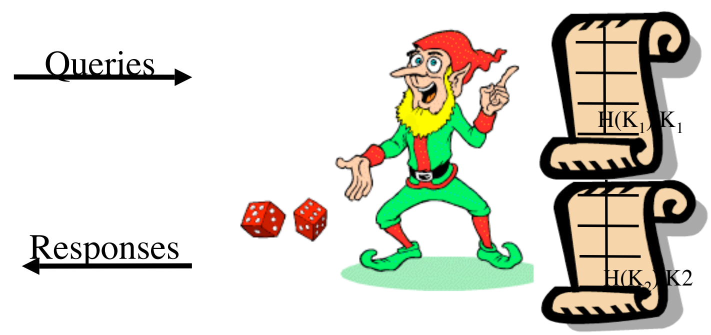
- Encryption
  - Suppose we encrypt M with Bob's public key
  - Only Bob's private key can decrypt to find M
- Digital Signature
  - **Sign** by "encrypting" with private key
  - Anyone can **verify** signature by "decrypting" with public key
  - But only private key holder could have signed
  - Like a handwritten signature

Topic Outline

- The Random Oracle model for Public Key Cryptosystems
  - Public key encryption and trapdoor one-way permutations
  - Digital signatures
- Looking under the hood
  - Knapsack
  - RSA
- Uses of Public Crypto
- The order of sign and encrypt

Public Key Encryption and Trap-door One-Way Permutation as Random Oracle

- Public Key Encryption Scheme:
  - Key pair (KR, KR^{-1}) generation function from random string R
    - $KR \rightarrow KR^{-1}$ is infeasible
  - $C = \{M)_{KR}$
  - $M = \{C)_{KR^{-1}}$



- In:
  - fixed size short string (plaintext) M,
  - Key KR
- Out: fixed size short string (ciphertext) C

Digital Signature as Random Oracle

- Public Key Signature Scheme:
  - Key pair (σR, VR) generation function
    - VR → σR is infeasible
  - $S = Sig_{\sigma R}(M)$
  - {True, False} = $Ver_{VR}(S)$

	Signing	Verifying
Input	Any string M + σR	S + VR
Output	S = hash(M) | cipher block	"True" or "False"

Looking Under the Hood

Knapsack

Knapsack Problem

- Given a set of n weights $W_0, W_1, \ldots, W_{n-1}$ and a sum S, is it possible to find $a_i \in \{0,1\}$ so that

$$S = a_0 W_0 + a_1 W_1 + \ldots + a_{n-1} W_{n-1}$$

(technically, this is "subset sum" problem)

- **Example**
  - Weights (62,93,26,52,166,48,91,141)
  - Problem: Find subset that sums to $S=302$
  - Answer: $62+26+166+48=302$
- The (general) knapsack is NP-complete

Knapsack Problem

- General knapsack (GK) is hard to solve
- But **super-increasing knapsack** (SIK) is easy
- SIK: each weight greater than the sum of all previous weights
- **SIK Example**
  - Weights (2,3,7,14,30,57,120,251)
  - Problem: Find subset that sums to S=186
  - Work from largest to smallest weight
  - Answer: 120+57+7+2=186

Knapsack Cryptosystem

1. Generate super-increasing knapsack (SIK)
2. Convert SIK into "general" knapsack (GK)
3. **Public Key:** GK
4. **Private Key:** SIK plus conversion factors

- Easy to encrypt with GK
- With private key, easy to decrypt (convert ciphertext to SIK)
- Without private key, must solve GK (???)

Knapsack Cryptosystem

- Let $(2,3,7,14,30,57,120,251)$ be the SIK
- Choose $m = 41$ and $n = 491$ with m, n relatively prime and n greater than sum of elements of SIK
- General knapsack

$$
\begin{aligned}
2 \cdot 41 \bmod 491 &= 82 \\
3 \cdot 41 \bmod 491 &= 123 \\
7 \cdot 41 \bmod 491 &= 287 \\
14 \cdot 41 \bmod 491 &= 83 \\
30 \cdot 41 \bmod 491 &= 248 \\
57 \cdot 41 \bmod 491 &= 373 \\
120 \cdot 41 \bmod 491 &= 10 \\
251 \cdot 41 \bmod 491 &= 471
\end{aligned}
$$

- General knapsack: $(82,123,287,83,248,373,10,471)$

Knapsack Example

- **Private key:** (2,3,7,14,30,57,120,251), n = 491, m^{-1}=12
  - $m^{-1} \bmod n = 41^{-1} \bmod 491 = 12$
  - $(x^{-1}\ x) \bmod n = 1 \bmod n$
- **Public key:** (82,123,287,83,248,373,10,471)
- **Throw away:** m = 41
- Example: Encrypt 150 = 10010110

  82 + 83 + 373 + 10 = 548 = C
- To decrypt,
  - $(C\ m^{-1}) \bmod n = (548 \cdot 12) \bmod 491 = 193 \bmod 491$
  - Solve (easy) SIK with S = 193
  - Obtain plaintext 10010110 = 150

Knapsack Weakness

- **Trapdoor:** Convert SIK into “general” knapsack using modular arithmetic
- **One-way:** General knapsack easy to encrypt, hard to solve; SIK easy to solve
- This knapsack cryptosystem is **insecure**
  - Broken by Shamir in 1983 with Apple II computer
  - The attack uses **lattice reduction**
- “General knapsack” is not general enough!
- This special knapsack is easy to solve!

RSA

Cocks (GCHQ), independently, by

Rivest, Shamir and Adleman (MIT)

basics

- Let p and q be two large prime numbers
- Let $N = pq$ be the **modulus**
- e relatively prime to $(p-1)(q-1)$ -- encryption exponent
- $d = e^{-1} \bmod (p-1)(q-1)$ -- decryption exponent
- **Throw Away:** p,q
- **Public key:** is (N, e),
- **Private key:** is d

encrypting & decrypting

- To encrypt message M compute
  - $C = M^e \bmod N$
- To decrypt C compute
  - $M = C^d \bmod N$
- Recall that e and N are public
- If attacker can factor N, he can use e to easily find d since $ed = 1 \bmod (p-1)(q-1)$
- Factoring the modulus breaks RSA
- It is not known whether factoring is the only way to break RSA

Simple RSA Example

- Select "large" primes $p = 11, q = 3$
- Then $N = pq = 33$ and $(p-1)(q-1) = 20$
- Choose $e = 3$ (relatively prime to 20)
- Find d such that $ed = 1 \bmod 20$, we find that $d = 7$ works
- **Public key:** $(N, e) = (33, 3)$
- **Private key:** $d = 7$

Simple RSA Example

- **Public key:** $(N, e) = (33, 3)$
- **Private key:** $d = 7$
- Suppose message $M = 8$
- Ciphertext C is computed as

$$C = M^e \bmod N = 8^3 = 512 = 17 \bmod 33$$

- Decrypt C to recover the message M by

$$M = C^d \bmod N = 17^7 = 410{,}338{,}673 = 12{,}434{,}505 * 33 + 8 = 8 \bmod 33$$

Uses for Public Key Crypto

Uses for Public Key Crypto

- Confidentiality
  - Transmitting data over insecure channel
  - Secure storage on insecure media
- Authentication
- Digital signature provides integrity and **non-repudiation**
  - No non-repudiation with symmetric keys

Non-non-repudiation

- Alice orders 100 shares of stock from Bob
- Alice computes **MAC** using symmetric key
- Stock drops, Alice claims she did not order
- Can Bob prove that Alice placed the order?
- **No!** Since Bob also knows symmetric key, he could have forged message
- **Problem:** Bob knows Alice placed the order, but he can't prove it

Non-repudiation

- Alice orders 100 shares of stock from Bob
- Alice **signs** order with her private key
- Stock drops, Alice claims she did not order
- Can Bob prove that Alice placed the order?
- **Yes!** Only someone with Alice's private key could have signed the order
- This assumes Alice's private key is not stolen (revocation problem)

Sign and Encrypt
vs
Encrypt and Sign

Public Key Notation

- **Sign** message M with Alice's **private key:** $[M]_{Alice}$
- **Encrypt** message M with Alice's **public key:** $\{M\}_{Alice}$
- Then

$$\{[M]_{Alice}\}_{Alice} = M$$

$$[\{M\}_{Alice}]_{Alice} = M$$

Confidentiality and Non-repudiation

- Suppose that we want confidentiality and non-repudiation
- Can public key crypto achieve both?
- Alice sends message to Bob
  - **Sign and encrypt** $\{[M]_{Alice}\}_{Bob}$
  - **Encrypt and sign** $[\{M\}_{Bob}]_{Alice}$
- Can the order possibly matter? (pg. 77-79 Stamp)

Sign and Encrypt

M = “I love you”

Alice

$\{[M]_{Alice}\}_{Bob}$ →

Bob

$\{[M]_{Alice}\}_{Charlie}$ →

Charlie

Q: What is the problem?

A: Charlie misunderstands crypto!

Encrypt and Sign

M = “My theory, which is mine, is this: ….”

Alice

$[\{M\}_{Bob}]_{Alice}$ →

Charlie

$[\{M\}_{Bob}]_{Charlie}$ →

Bob

Note that Charlie cannot decrypt M

Q: What is the problem?

A: Bob misunderstands crypto!

Summary

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- Looking under the hood
  - Knapsack
  - RSA
- Uses of Public Crypto
- The order of sign and encrypt