Public Key Cryptography

EECE 412

What is it?

- Two keys
  - Sender uses recipient's **public key** to encrypt
  - Receiver uses his **private key** to decrypt
- Based on **trap door, one way function**
  - Easy to compute in one direction
  - Hard to compute in other direction
  - "Trap door" used to create keys
  - Example: Given p and q, product $N=pq$ is easy to compute, but given N, it is hard to find p and q

How is it used?

Encryption

Suppose we encrypt M with Bob's public key

Only Bob's private key can decrypt to find M

Digital Signature

- **Sign** by "encrypting" with private key

Anyone can **verify** signature by "decrypting" with public key

But only private key holder could have signed

Like a handwritten signature

Topic Outline

- The Random Oracle model for Public Key Cryptosystems
  - Public key encryption and trapdoor one-way permutations
  - Digital signatures
- Looking under the hood
  - Knapsack
  - RSA
- Uses of Public Crypto
- The order of sign and encrypt

Public Key Encryption and Trap-door One-Way Permutation as Random Oracle

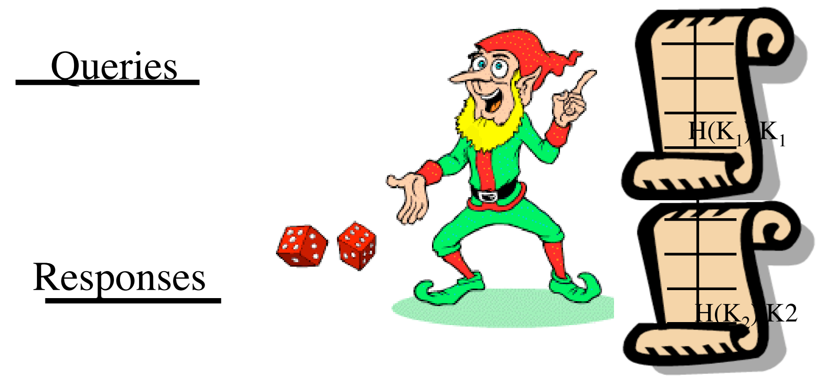
Public Key Encryption Scheme:

Key pair (KR, KR^{-1}) generation function from random string R

$KR \rightarrow KR^{-1}$ is infeasible

- $C = \{M\}_{KR}$

$M = \{C\}_{KR^{-1}}$



In:

fixed size short string (plaintext) M,

Key KR

Out: fixed size short string (ciphertext) C

Digital Signature as Random Oracle

Public Key Signature Scheme:

Key pair (σR, VR) generation function

VR → σR is infeasible

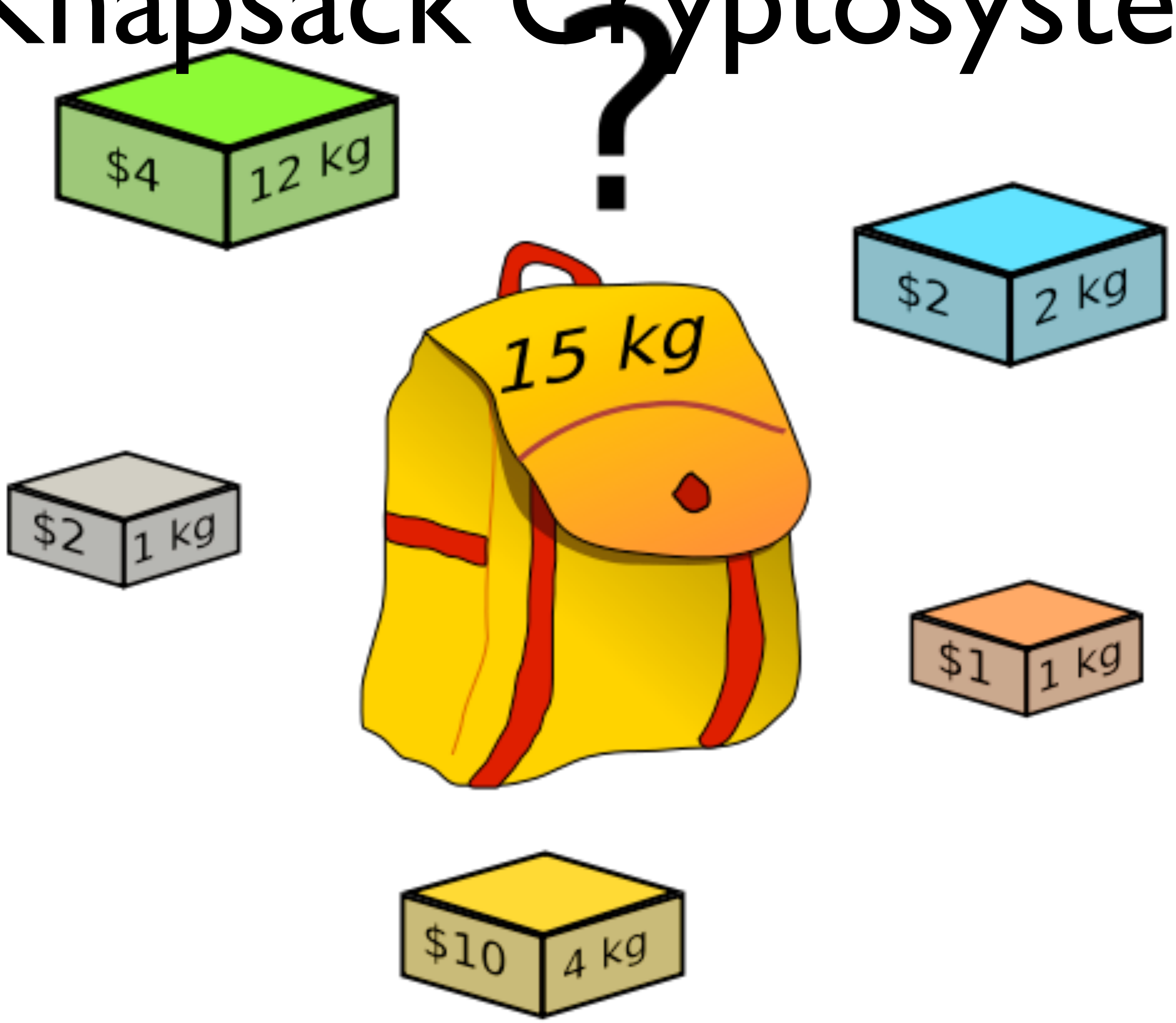
$S = Sig_{\sigma R}(M)$

{True, False} = $Ver_{VR}(S)$

	Signing	Verifying
Input	Any string M + σR	S + VR
Output	S = hash(M) | cipher block	"True" or "False"

Looking Under the Hood

Knapsack Cryptosystem



Knapsack Problem

Given a set of n weights $W_0, W_1, \ldots, W_{n-1}$ and a sum S, is it possible to find $a_i \in \{0,1\}$ so that

$$S = a_0 W_0 + a_1 W_1 + \ldots + a_{n-1} W_{n-1}$$

(technically, this is "subset sum" problem)

Example

Weights $(62,93,26,52,166,48,91,141)$

Problem: Find subset that sums to $S=302$

Answer: $62+26+166+48=302$

The (general) knapsack is NP-complete

Knapsack Problem

General knapsack (GK) is hard to solve

But **super-increasing knapsack** (SIK) is easy

SIK: each weight greater than the sum of all previous weights

- **SIK Example**

  Weights (2,3,7,14,30,57,120,251)

  Problem: Find subset that sums to S=186

  Work from largest to smallest weight

  Answer: 120+57+7+2=186

Knapsack Cryptosystem

1. Generate super-increasing knapsack (SIK)
2. Convert SIK into "general" knapsack (GK)
3. **Public Key:** GK
4. **Private Key:** SIK plus conversion factors

- Easy to encrypt with GK
- With private key, easy to decrypt (convert ciphertext to SIK)
- Without private key, must solve GK (???)

Knapsack Cryptosystem

- Let $(2,3,7,14,30,57,120,251)$ be the SIK
- Choose $\mathrm{m} = 41$ and $\mathrm{n} = 491$ with m, n relatively prime and n greater than sum of elements of SIK
- General knapsack

$$(2 \cdot 41) \bmod 491 = 82$$

$$3 \cdot 41 \bmod 491 = 123$$

$$7 \cdot 41 \bmod 491 = 287$$

$$14 \cdot 41 \bmod 491 = 83$$

$$30 \cdot 41 \bmod 491 = 248$$

$$57 \cdot 41 \bmod 491 = 373$$

$$120 \cdot 41 \bmod 491 = 10$$

$$251 \cdot 41 \bmod 491 = 471$$

- General knapsack: $(82,123,287,83,248,373,10,471)$

Knapsack Example

Private key: (2,3,7,14,30,57,120,251), n = 491, m^{-1}=12

- m^{-1} mod n = 41^{-1} mod 491 = 12
- (x^{-1} x) mod n = 1 mod n

Public key: (82,123,287,83,248,373,10,471)

Throw away: m = 41

Example: Encrypt 150 = 10010110

82 + 83 + 373 + 10 = 548 = C

To decrypt,

(C m^{-1}) mod n = (548 · 12) mod 491 = 193 mod 491

Solve (easy) SIK with S = 193

Obtain plaintext 10010110 = 150

Knapsack Weakness

Trapdoor: Convert SIK into "general" knapsack using modular arithmetic

One-way: General knapsack easy to encrypt, hard to solve; SIK easy to solve

This knapsack cryptosystem is **insecure**

- Broken by Shamir in 1983 with Apple II computer
  - The attack uses **lattice reduction**

"General knapsack" is not general enough!

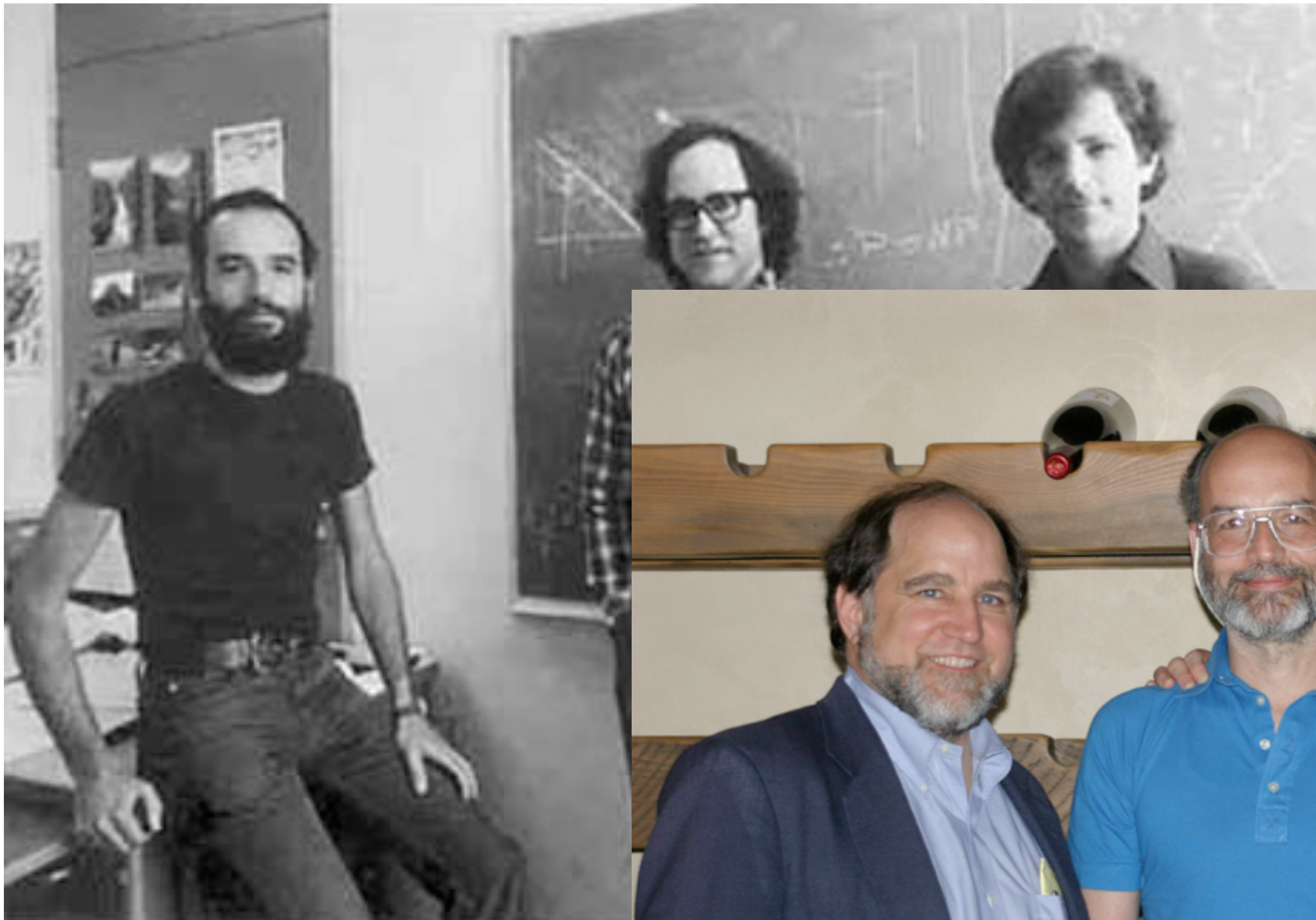
This special knapsack is easy to solve!

RSA

Cocks (GCHQ), independently, by

Rivest, Shamir and Adleman (MIT)

Rivest, Shamir, and Adleman



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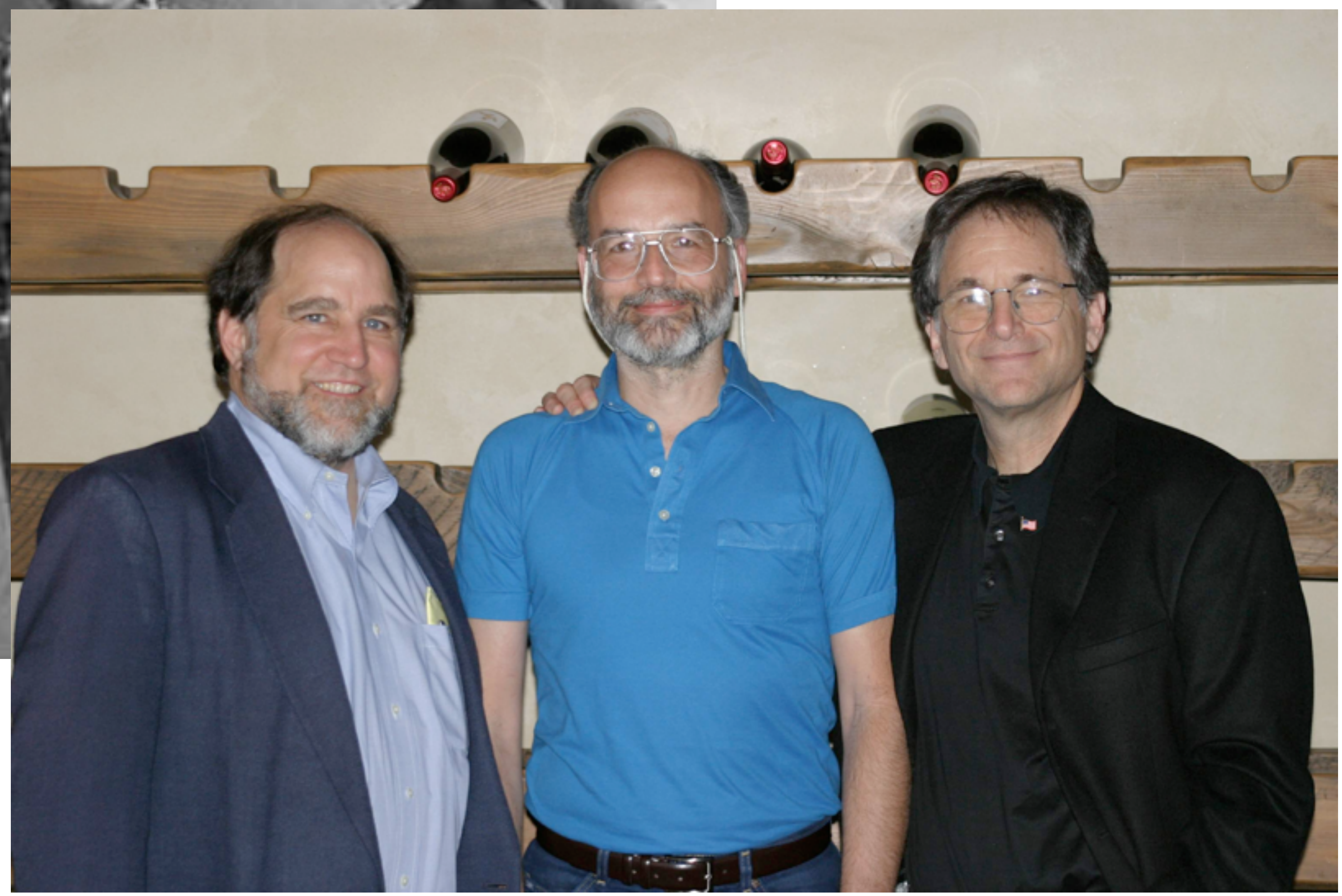

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basics

Let p and q be two large (e.g., 200 digits) prime numbers

use probabilistic primality tests to find p & q quickly

Let $n = p \times q$ be the modulus

Factoring n is supposed to be hard (i.e., billions of years)

e relatively prime to $(p-1)(q-1)$ -- encryption exponent

$d = e^{-1} \bmod (p-1)(q-1)$ -- decryption exponent

Throw Away: p, q

- **Public key:** (n, e)

Private key: d

Notation: public is in cyan, secret is in red

encrypting & decrypting

To encrypt message M compute

- $C = M^e \bmod n$ -- fast with modular exponentiation

To decrypt C compute

- $M = C^d \bmod n$

Recall that e and n are public

If attacker can factor n, he can use e to easily find d since $ed = 1 \bmod (p-1)(q-1)$

Factoring the modulus breaks RSA

It is not known whether factoring is the only way to break RSA

RSA in the works

simple RSA example: initialization

Select "large" primes $p = 43, q = 59$

Then $n = p \times q = 2537$ and $(p-1)(q-1) = 2436$

Choose $e = 13$ (relatively prime to 2436)

Find d such that $ed = 1 \bmod (p-1)(q-1)$,
we find that $d = 937$ works

note: d exists because $\gcd(e, (p-1)(q-1)) = 1$

Public key: $(N, e) = (2537, 13)$

Private key: $d = 937$

simple RSA example: encryption

plain text: M = “STOP” = (18 19, 14 15)

ciphertext: C = M^e mod n =
(1819^{13} mod 2537, 1415^{13} mod 2537) =
20 81 21 82 = “UDVE”

fast modular exponentiation

simple RSA example: decryption

ed = 1 mod (p–1)(q–1) ⇒ ∃ k s.t. ed = k(p-1)(q-1)+1

$C^d \equiv (M^e)^d = M^{de} = M^{1+ k(p-1)(q-1)} (\text{mod } n)$

$M^{p-1} \equiv 1 \text{ mod } p$ and $M^{q-1} \equiv 1 \text{ mod } q$
by Fermat's Little Theorem:

> If *p* is prime and *a* is an integer not divisible by *p*, then $a^{p-1} \equiv 1 \text{ mod } p$. Furthermore $a^p \equiv a \text{ mod } p$

$C^d \equiv M \times ((M^{p-1})^{k(q-1)}) \text{ mod } p \equiv M \times 1 \text{ mod } p \equiv M \text{ mod } p$

$C^d \equiv M \times ((M^{q-1})^{k(p-1)}) \text{ mod } p \equiv M \times 1 \text{ mod } p \equiv M \text{ mod } q$

Because gcd(p,q) =1, $C^d \equiv M \text{ mod } p \times q$ by Chinese Remainder Theorem

simple RSA example: decryption

Decrypt message 0981 0461

$M \equiv C^d \bmod p \times q$

$0981^{937} \bmod 2537 = 0704$ = “HE”

$0461^{937} \bmod 2537 = 1115$ = “LP”

HELP

Uses for Public Key Crypto

Uses for Public Key Crypto

- Confidentiality
  - Transmitting data over insecure channel
  - Secure storage on insecure media
- Authentication
- Digital signature provides integrity and **non-repudiation**
  - No non-repudiation with symmetric keys

Non-non-repudiation

Alice orders 100 shares of stock from Bob

Alice computes **MAC** using symmetric key

Stock drops, Alice claims she did not order

Can Bob prove that Alice placed the order?

No! Since Bob also knows symmetric key, he could have forged message

Problem: Bob knows Alice placed the order, but he can't prove it

Non-repudiation

Alice orders 100 shares of stock from Bob

Alice **signs** order with her private key

Stock drops, Alice claims she did not order

Can Bob prove that Alice placed the order?

Yes! Only someone with Alice's private key could have signed the order

This assumes Alice's private key is not stolen (revocation problem)

Sign and Encrypt
vs
Encrypt and Sign

Public Key Notation

Sign message M with Alice's **private key:** $[M]_{Alice}$

Encrypt message M with Alice's **public key:** $\{M\}_{Alice}$

Then

$$\{[M]_{Alice}\}_{Alice} = M$$

$$[\{M\}_{Alice}]_{Alice} = M$$

Confidentiality and Non-repudiation

Suppose that we want confidentiality and non-repudiation

Can public key crypto achieve both?

Alice sends message to Bob

- **Sign and encrypt** $\{[M]_{Alice}\}_{Bob}$
- **Encrypt and sign** $[\{M\}_{Bob}]_{Alice}$

Can the order possibly matter? (see Stamp)

Sign and Encrypt

M = "I love you"

$\{[M]_{Alice}\}_{Bob}$

$\{[M]_{Alice}\}_{Charlie}$

Alice

Bob

Charlie

Q: What is the problem?

A: Charlie misunderstands crypto!

Encrypt and Sign

M = “My theory, which is mine, is this: ….”

Alice

$[\{M\}_{Bob}]_{Alice}$

Charlie

$[\{M\}_{Bob}]_{Charlie}$

Bob

Note that Charlie cannot decrypt M

Q: What is the problem?

A: Bob misunderstands crypto!

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Summary

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- Looking under the hood
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- The order of sign and encrypt