AA278A Lecture Notes 6
Spring 2005
Stability of Hybrid Systems

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April 21, 2005

Figure 1: Aleksandr Mikhailovich Lyapunov, 1857 - 1918

For the purpose of studying stability of hybrid automata, we will again drop reference to inputs (and outputs) of the system and focus on the state trajectory. Later, we will bring the inputs (and outputs) back in when we talk about stabilizing controllers.

1 Review of Stability for Continuous Systems

For reference to the following material, see [1, 2], Chapters 3 and 2 respectively.

Consider the following continuous system:

$$\dot{x} = f(x), \qquad x(0) = x_0 \tag{1}$$

where $f : \mathbb{R}^n \to \mathbb{R}^n$ is globally Lipschitz continuous.

- **Definition (Equilibrium of (1)):** $x = x_e$ is an *equilibrium point* of (1) if $f(x_e) = 0$. Without loss of generality in the following we will assume that $x_e = 0$.

- **Definition (Stability of (1)):** The equilibrium point $x_e = 0$ of (1) is *stable* (in the sense of Lyapunov) if for all $\epsilon > 0$ there exists a $\delta > 0$ such that

  $$||x_0|| < \delta \Rightarrow ||x(t)|| < \epsilon, \forall t \geq 0 \tag{2}$$

  where $x : [0, \infty) \to \mathbb{R}^n$ is the solution to (1), starting at x_0.

  The equilibrium point $x_e = 0$ of (1) is *asymptotically stable* if it is stable and δ can be chosen so that

  $$||x_0|| < \delta \Rightarrow \lim_{t\to\infty} ||x(t)|| = 0 \tag{3}$$

- More definitions for stability: exponentially stable, globally (asymptotically, exponentially) stable, locally (asymptotically, exponentially) stable, unstable ...

- Note that the definition of stability allows $x_e = 0$ to be stable without $x(t)$ converging to 0; note also that for system (1) with a single unstable equilibrium point, for all x_0 the solution can be bounded.

Consider a continuously differentiable (C^1) function $V : \mathbb{R}^n \to \mathbb{R}$. The rate of change of V along solutions of (1) is computed as:

$$\dot{V}(x) = \sum_{i=1}^{n} \frac{\partial V}{\partial x_i}\dot{x}_i = \sum_{i=1}^{n} \frac{\partial V}{\partial x_i} f_i(x) = \frac{\partial V}{\partial x} f(x) \tag{4}$$

This function is denoted the *Lie derivative* of V with respect to f.

Theorem 1 (Lyapunov Stability Theorem) *Let $x_e = 0$ be an equilibrium point of $\dot{x} = f(x), x(0) = x_0$ and $D \subset \mathbb{R}^n$ a set containing $x_e = 0$. If $V : D \to \mathbb{R}$ is a C^1 function such that*

$$V(0) = 0 \tag{5}$$
$$V(x) > 0, \forall x \in D \backslash \{0\} \tag{6}$$
$$\dot{V}(x) \leq 0, \forall x \in D \tag{7}$$

then x_e is stable. Furthermore, if $x_e = 0$ is stable and

$$\dot{V}(x) < 0, \forall x \in D \backslash \{0\} \tag{8}$$

then x_e is asymptotically stable.

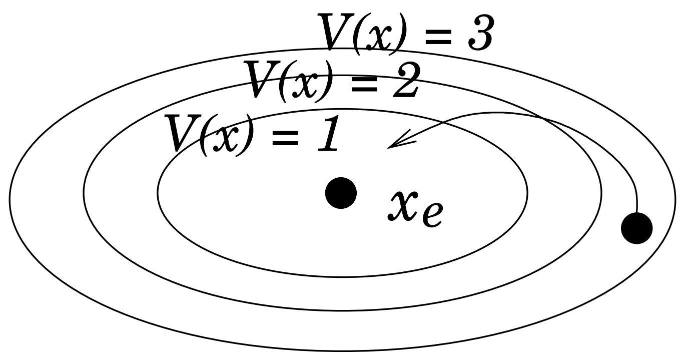

Figure 2: Level sets $V(x) = 1$, $V(x) = 2$, and $V(x) = 3$ for a Lyapunov function V; thus if a state trajectory enters one of these sets, it has to stay inside it since $\dot{V}(x) \leq 0$.

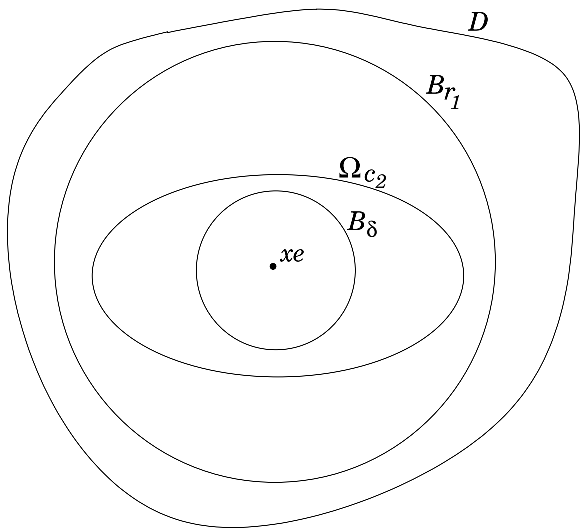

Figure 3: Figure for Proof of Lyapunov Stability Theorem (for continuous systems); WLOG $x_e = 0$.

Note that the Lyapunov function defines level sets $\{x \in \mathbb{R}^n : V(x) \leq c\}$ for $c > 0$ (see Figure 2). If a state trajectory enters one of these sets, it has to stay inside it, since $\dot{V}(x) \leq 0$ implies that if $V(x) = c$ at $t = t_0$, then $V(x(t)) \leq V(x(0)) \leq c$, $\forall t \geq t_0$.

Proof: For stability, we need to prove that for all $\epsilon > 0$ there exists $\delta > 0$ such that:

$$||x_0|| < \delta \Rightarrow ||x(t)|| < \epsilon, \forall t \geq 0 \tag{9}$$

We will use the following notation: for any $r > 0$, let $B_r = \{x \in \mathbb{R}^n : ||x|| < r\}$, $S_r = \{x \in \mathbb{R}^n : ||x|| = r\}$, and $\Omega_r = \{x \in \mathbb{R}^n : V(x) < r\}$.

See Figure 3. Choose $r_1 \in (0, \epsilon)$ such that $B_{r_1} \subseteq D$ (we do this because there is no guarantee that $B_\epsilon \subseteq D$). Let $c_1 = \min_{x \in S_{r_1}} V(x)$. Choose $c_2 \in (0, c_1)$. Note that there is no guarantee that $\Omega_{c_2} \subset B_{r_1}$. *Why not?* However, if $\delta > 0$ is chosen so that $B_\delta \subseteq \Omega_{c_2}$, then $V(x_0) < c_2$. Since V is non-increasing along system executions, executions that start inside B_δ cannot leave Ω_{c_2}. Thus for all $t > 0$ we have $x(t) \in B_{r_1} \subset B_\epsilon$. Thus $||x(t)|| \leq \epsilon$ for all $t > 0$. ∎

- **Example (Pendulum)** Consider the pendulum, unit mass, unit length, where x_1 is the angle of the pendulum with the vertical, and x_2 is the angular velocity of the pendulum.

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -g \sin x_1 \end{aligned}$$

To show that $x_e = 0$ (pendulum pointing downwards) is a stable equilibrium, consider the candidate Lyapunov function:

$$V(x) = g(1 - \cos x_1) + \frac{x_2^2}{2} \tag{10}$$

defined over the set $\{x \in \mathbb{R}^n : -\pi < x_1 < \pi\}$. Clearly, $V(0) = 0$, and $V(x) > 0, \forall x \in \{x \in \mathbb{R}^n : -\pi < x_1 < \pi\} \backslash \{0\}$. Also,

$$\dot{V}(x) = [g \sin x_1 \;\; x_2] \begin{bmatrix} x_2 \\ -g \sin x_1 \end{bmatrix} = 0$$

so the equilibrium point $x_e = 0$ is stable. Is it asymptotically stable?

- Finding Lyapunov functions in general is HARD. Often a solution is to try to use the *energy* of the system as a Lyapunov function (as in the example above). However, for linear systems, finding Lyapunov functions is easy: For a stable linear system $\dot{x} = Ax$, a Lyapunov function is given by $V(x) = x^T P x$, where P is a positive definite symmetric matrix which solves the *Lyapunov Equation* $A^T P + PA = -I$. (Recall that a matrix P is said to be positive definite if $x^T P x > 0$ for all $x \neq 0$. It is called positive semidefinite if $x^T P x \geq 0$ for all $x \neq 0$.)

2 Stability of Hybrid Systems

Consider an autonomous hybrid automaton $H = (S, \text{Init}, f, \text{Dom}, R)$.

Definition 2 (Equilibrium of a Hybrid Automaton) *The continuous state $x_e = 0 \in \mathbb{R}^n$ is an equilibrium point of H if:*

1. *$f(q, 0) = 0$ for all $q \in \mathbf{Q}$, and*
2. *$R(q, 0) \subset Q \times 0$.*

- Thus, discrete transitions are allowed out of $(q, 0)$, as long as the system jumps to a (q', x) in which $x = x_e = 0$.

- It follows from the above definition that if $(q_0, 0) \in \text{Init}$ and $(\tau, (q, x))$ represents the hybrid execution starting at $(q_0, 0)$, then $x(t) = 0$ for all $t \in \tau$.

As we did for continuous systems, we would like to characterize the notion that if the continuous state starts close to the equilibrium point, it stays close, or converges, to it.

- **Definition (Stable Equilibrium of a Hybrid Automaton):** Let $x_e = 0$ be an equilibrium point of the hybrid automaton H. Then $x_e = 0$ is stable if for all $\epsilon > 0$ there exists $\delta > 0$ such that for all $(\tau, (q, x))$ starting at (q_0, x_0),

$$||x_0|| < \delta \Rightarrow ||x(t)|| < \epsilon, \forall t \in \tau \tag{11}$$

- **Definition (Asymptotically Stable Equilibrium of a Hybrid Automaton):** Let $x_e = 0 \in X$ be an equilibrium point of the hybrid automaton H. Then $x_e = 0$ is asymptotically stable if it is stable and δ can be chosen so that for all $(\tau, (q, x))$ starting at (q_0, x_0),

$$||x_0|| < \delta \Rightarrow \lim_{t \to \tau_\infty} ||x(t)|| = 0 \tag{12}$$

- **Remark:** In the above, $(\tau, (q, x))$ is considered to be an infinite execution, with $\tau_\infty = \sum_i (\tau_i' - \tau_i)$. Notice that $\tau_\infty < \infty$ if χ is Zeno and $\tau_\infty = \infty$ otherwise.

One would expect that a hybrid system for which each discrete state's continuous system is stable would be stable, at least if $R(q, x) \in Q \times \{x\}$ for all x. But this is NOT NECESSARILY the case:

- **Example:** Consider the hybrid automaton H with:
  - $Q = \{q_1, q_2\}$, $X = \mathbb{R}^2$
  - Init $= Q \times \{x \in X : ||x|| > 0\}$
  - $f(q_1, x) = A_1 x$ and $f(q_2, x) = A_2 x$, with:

$$A_1 = \begin{bmatrix} -1 & 10 \\ -100 & -1 \end{bmatrix}, A_2 = \begin{bmatrix} -1 & 100 \\ -10 & -1 \end{bmatrix}$$

  - Dom $= \{q_1, \{x \in \mathbb{R}^2 : x_1 x_2 \leq 0\}\} \cup \{q_2, \{x \in \mathbb{R}^2 : x_1 x_2 \geq 0\}\}$
  - $R(q_1, \{x \in \mathbb{R}^2 : x_1 x_2 \geq 0\}) = (q_2, x)$ and $R(q_2, \{x \in \mathbb{R}^2 : x_1 x_2 \leq 0\}) = (q_1, x)$

- **Remark 1:** Since $f(q_1, 0) = f(q_2, 0) = 0$ and $R(q_1, 0) = (q_2, 0), R(q_2, 0) = (q_1, 0)$, $x_e = 0$ is an equilibrium of H.

- **Remark 2:** Since the eigenvalues of both systems are at $-1 \pm j\sqrt{1000}$, the continuous systems $\dot{z} = A_i x$ for $i = 1, 2$ are asymptotically stable. (See phase portraits for each in Figure 4.)

- **Remark 3:** $x_e = 0$ is unstable for H! If the switching is flipped, then $x_e = 0$ is stable! (See phase portraits for each in Figure 5.)

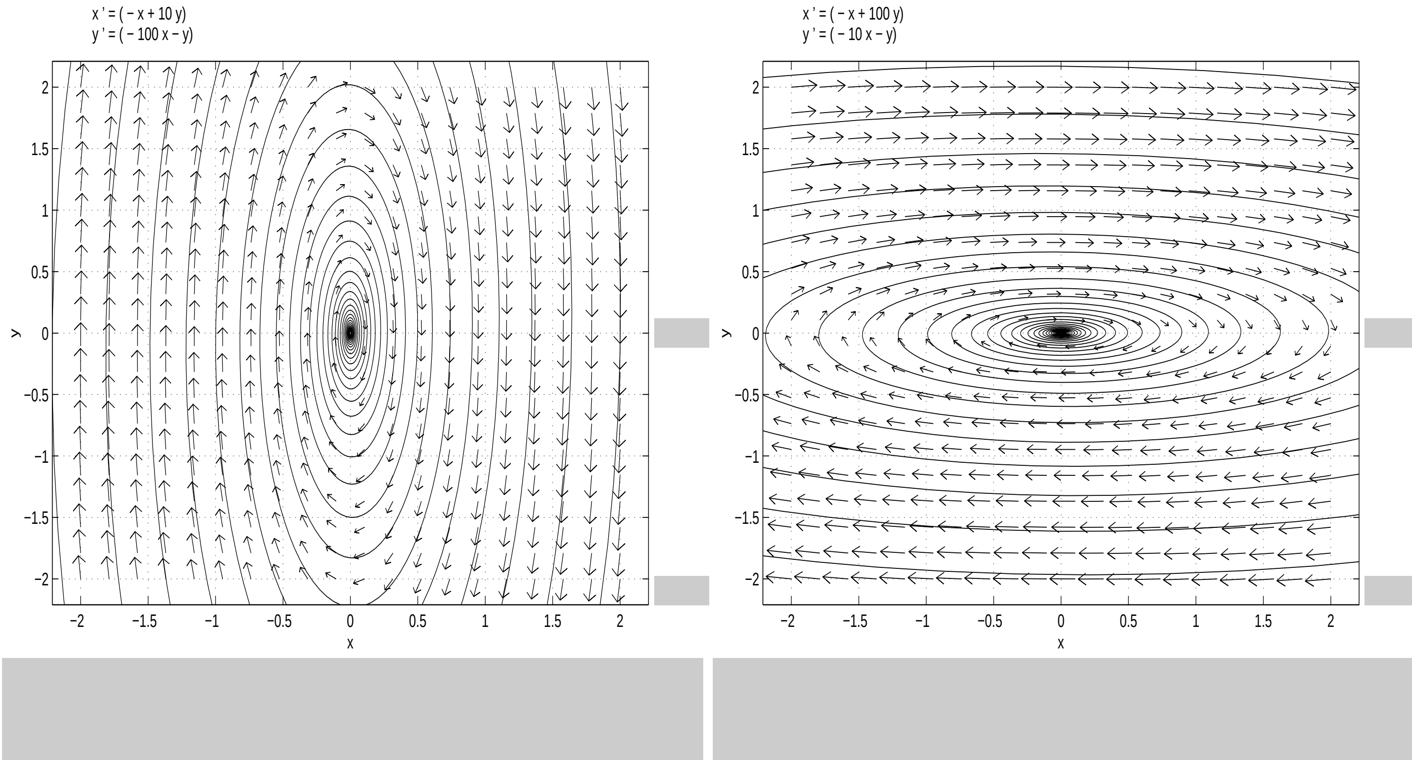

Figure 4: (a) Phase portrait of $\dot{x} = A_1 x$; (b) Phase portrait of $\dot{x} = A_2 x$. Figure generated using phase plane software from `http://math.rice.edu/~polking/odesoft/`, freely downloadable.

- **Remark 4:** This simple example (drawn from [3]) shows that in general we cannot expect to analyze the stability of a hybrid system by studying the continuous components separately.

Theorem 3 (Lyapunov Stability Theorem (for hybrid systems)) *Consider a hybrid automaton H with $x_e = 0$ an equilibrium point, and $R(q,x) \in Q \times \{x\}$. Assume that there exists an open set $D \subset Q \times \mathbb{R}^n$ such that $(q,0) \in D$ for some $q \in Q$. Let $V : D \to \mathbb{R}$ be a C^1 function in its second argument such that for all $q \in Q$:*

1. *$V(q,0) = 0$;*
2. *$V(q,x) > 0$ for all x, $(q,x) \in D \backslash \{0\}$, and*
3. *$\frac{\partial V(q,x)}{\partial x} f(q,x) \leq 0$ for all x, $(q,x) \in D$*

If for all (τ, q, x) starting at (q_0, x_0) where $(q_0, x_0) \in \mathrm{Init} \cap D$, and all $q' \in Q$, the sequence $\{V(q(\tau_i), x(\tau_i)) : q(\tau_i) = q'\}$ is non-increasing (or empty), then $x_e = 0$ is a stable equilibrium of H.

- We call such function "Lyapunov-like" (see Figure 6).

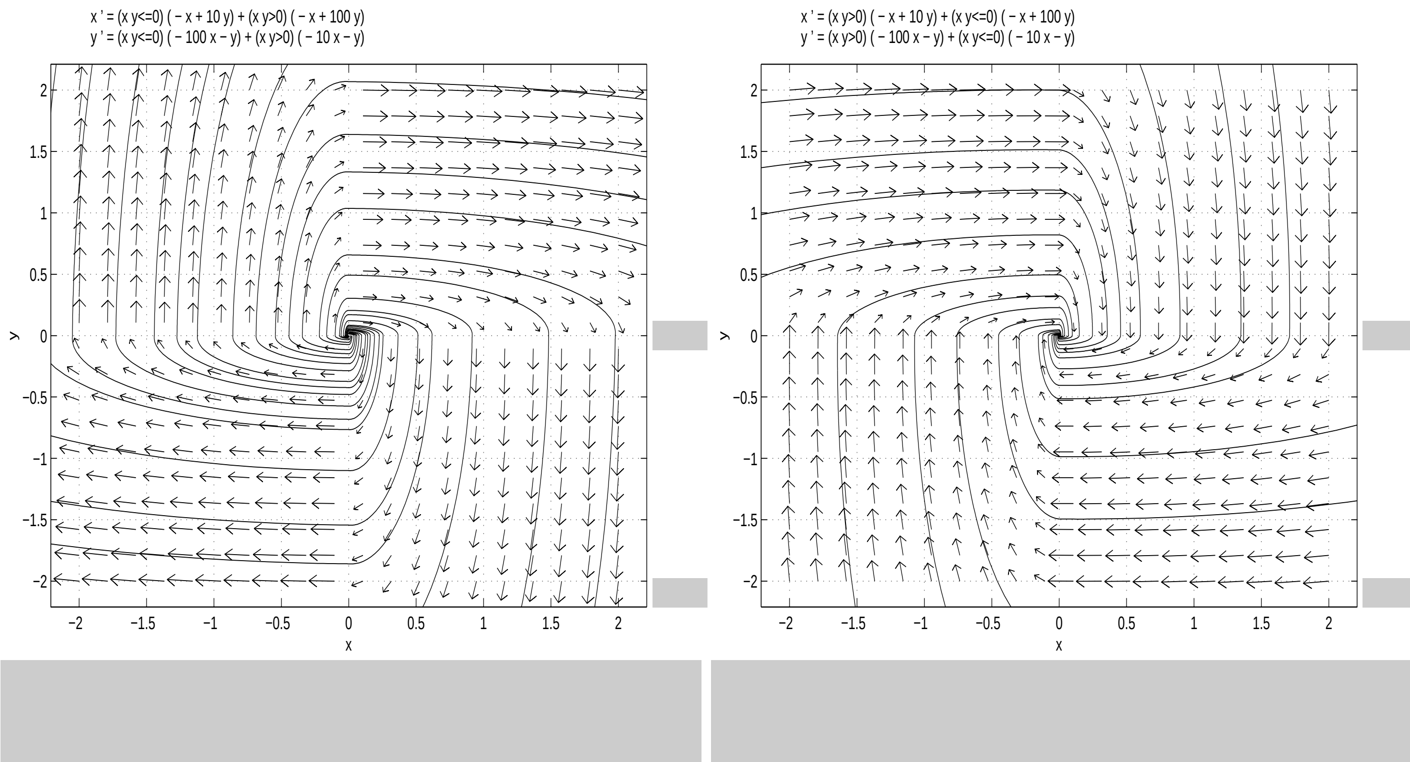

Figure 5: (a) Phase portrait of H; (b) Phase portrait of H, switching conditions flipped.

- A drawback of this Theorem is that the sequence $\{V(q(\tau_i), x(\tau_i))\}$ must be evaluated (which may require integrating the vector field and then we lose the fundamental advantage of Lyapunov theory)
- Also, it is in general difficult to derive such a function V
- HOWEVER, for certain classes of hybrid automata, which have vector fields linear in x, computationally attractive methods exist to derive Lyapunov-like functions V

Proof: We sketch the proof for $Q = \{q_1, q_2\}$ and $(q, \cdot) \notin R(q, \cdot)$. Define the sets in Figure 7 similar to the previous proof, ie.

$$\Omega_{c_{2_i}} = \{x \in B_{r_{1_i}} : V(q_i, x) < c_{2_i}\} \tag{13}$$

where $c_{2_i} \in (0, c_{1_i})$ where $c_{1_i} = \min_{x \in S_{r_{1_i}}} V(q_i, x)$. Now let $r = \min\{\delta_1, \delta_2\}$, and inside B_r, in each of q_1 and q_2, repeat the construction above, ie.

$$\Omega_{\overline{c}_{2_i}} = \{x \in B_r : V(q_i, x) < \overline{c}_{2_i}\} \tag{14}$$

where $\overline{c}_{2_i} \in (0, \min_{x \in S_r} V(q_i, x))$. Also, let $B_{\overline{\delta}_i} \subset \Omega_{\overline{c}_{2_i}}$. Let $\delta = \min\{\overline{\delta}_1, \overline{\delta}_2\}$. Consider the hybrid trajectory (τ, q, x) starting at $x_0 \in B_\delta$, and assume that the initial discrete state q_0 is equal to q_1. By the corresponding continuous Lyapunov theorem, $x(t) \in \Omega_{\overline{c}_{2_1}}$ for $t \in [\tau_0, \tau_0']$. Therefore, $x(\tau_1) = x(\tau_0') \in \Omega_{\overline{c}_{2_2}}$ (where equality holds because of the restricted

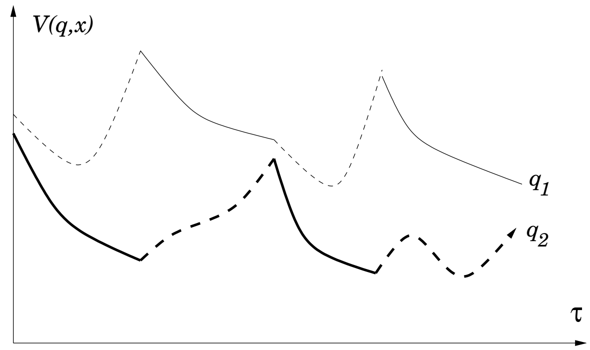

Figure 6: Showing $V(q_1, x)$ and $V(q_2, x)$. Solid segments on q_i mean that the system is in q_i at that time, dotted segments mean the system is in q_j, $j \neq i$.

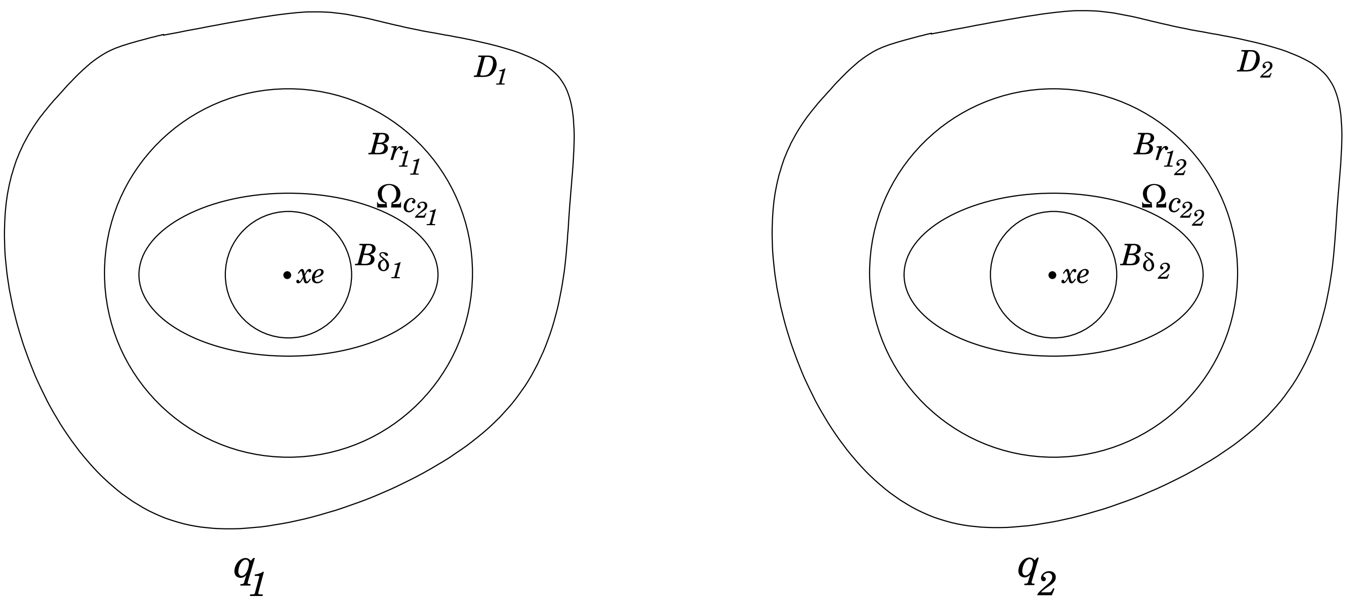

Figure 7: Figure for Proof of Lyapunov Stability Theorem (for hybrid systems).

definition of the transition map). By the continuous Lyapunov Theorem again, $x(t) \in \Omega_{\bar{c}_{2_2}}$ and thus $x(t) \in B_\epsilon$ for $t \in [\tau_1, \tau_1']$. By the assumption of the non-increasing sequence, $x(\tau_1') = x(\tau_2) \in \Omega_{\bar{c}_{2_1}}$. The result follows by induction. ∎

Corollary 4 (A more restrictive Lyapunov Stability Theorem (for hybrid systems)) *Consider a hybrid automaton H with $x_e = 0$ an equilibrium point, and $R(q, x) \in Q \times \{x\}$. Assume that there exists an open set $D \subset \mathbb{R}^n$ such that $0 \in D$. Let $V : D \to \mathbb{R}$ be a C^1 function such that for all $q \in Q$:*

1. *$V(0) = 0$;*
2. *$V(x) > 0$ for all $x \in D \backslash \{0\}$, and*
3. *$\frac{\partial V(x)}{\partial x} f(q, x) \leq 0$ for all $x \in D$*

Then $x_e = 0$ is a stable equilibrium of H.

Proof: Define $\hat{V} : \mathbf{Q} \times \mathbb{R}^n \to \mathbb{R}$ by $\hat{V}(q, x) = V(x)$ for all $q \in \mathbf{Q}$, $x \in \mathbb{R}^n$ and apply Theorem 3. ∎

3 Lyapunov Stability for Piecewise Linear Systems

Theorem 5 (Lyapunov Stability for Linear Systems) *The equilibrium point $x_e = 0$ of $\dot{x} = Ax$ is asymptotically stable if and only if for all matrices $Q = Q^T > 0$ there exists a unique matrix $P = P^T > 0$ such that*

$$PA + A^T P = -Q \tag{15}$$

Proof: For the "if" part of the proof, consider the Lyapunov function (from Lecture Notes 4) $V(x) = x^T P x$. Thus,

$$\dot{V} = x^T P \dot{x} + \dot{x}^T P x = x^T (PA + A^T P) x) = -x^T Q x < 0 \tag{16}$$

For the "only if" part of the proof, consider

$$P = \int_0^\infty e^{A^T t} Q e^{At} dt \tag{17}$$

which is well-defined since $\mathrm{Re}(\lambda_i(A)) < 0$. Clearly,

$$\begin{aligned} PA + A^T P &= \int_0^\infty e^{A^T t} Q e^{At} A dt + \int_0^\infty A^T e^{A^T t} Q e^{At} dt & (18) \\ &= \int_0^\infty \frac{d}{dt} e^{A^T t} Q e^{At} dt = -Q & (19) \end{aligned}$$

Also, P is unique: to prove this, assume there exists another solution $\hat{P} \neq P$. Then,

$$\begin{aligned} 0 &= e^{A^T t}(Q - Q) e^{At} & (20) \\ &= e^{A^T t}[(P - \hat{P})A + A^T(P - \hat{P})] e^{At} & (21) \\ &= \frac{d}{dt} e^{A^T t}(P - \hat{P}) e^{At} & (22) \end{aligned}$$

which means that $e^{A^T t}(P - \hat{P}) e^{At}$ is constant for all $t \geq 0$. Thus,

$$e^{A^T 0}(P - \hat{P}) e^{A0} = \lim_{t \to \infty} e^{A^T t}(P - \hat{P}) e^{At} \tag{23}$$

thus $P = \hat{P}$, which contradicts the assumption and thus concludes the proof. ∎

- Equation (15) is called a Lyapunov equation. We may also write the matrix condition in the Theorem of Lyapunov Stability for Linear Systems as the following inequality:

$$A^T P + PA < 0 \tag{24}$$

which is called a *linear matrix inequality* (LMI), since the left hand side is linear in the unknown P.

Example: Switched Linear System, Revisited.

Consider the linear hybrid system example from page 5 (with the switching flipped):

- $Q = \{q_1, q_2\}$, $X = \mathbb{R}^2$
- $\text{Init} = Q \times \{x \in X : ||x|| > 0\}$
- $f(q_1, x) = A_1 x$ and $f(q_2, x) = A_2 x$, with:

$$A_1 = \begin{bmatrix} -1 & 10 \\ -100 & -1 \end{bmatrix}, A_2 = \begin{bmatrix} -1 & 100 \\ -10 & -1 \end{bmatrix}$$

- $\text{Dom} = \{q_1, \{x \in \mathbb{R}^2 : x_1 x_2 \geq 0\}\} \cup \{q_2, \{x \in \mathbb{R}^2 : x_1 x_2 \leq 0\}\}$
- $R(q_1, \{x \in \mathbb{R}^2 : x_1 x_2 \leq 0\}) = (q_2, x)$ and $R(q_2, \{x \in \mathbb{R}^2 : x_1 x_2 \geq 0\}) = (q_1, x)$

Proposition 6 *$x = 0$ is an equilibrium of H.*

Proof: $f(q_1, 0) = f(q_2, 0) = 0$ and $R(q_1, 0) = (q_2, 0), R(q_2, 0) = (q_1, 0)$. ∎

Proposition 7 *The continuous systems $\dot{x} = A_i x$ for $i = 1, 2$ are asymptotically stable.*

Recall that $P_i > 0$ (positive definite) if and only if $x^T P_i x > 0$ for all $x \neq 0$, and I is the identity matrix.

Consider the candidate Lyapunov function:

$$V(q, x) = \begin{cases} x^T P_1 x & \text{if } q = q_1 \\ x^T P_2 x & \text{if } q = q_2 \end{cases}$$

Check that the conditions of the Theorem hold. For all $q \in \mathbf{Q}$:

1. $V(q, 0) = 0$
2. $V(q, x) > 0$ for all $x \neq 0$ (since the P_i are positive definite)

3. $\frac{\partial V}{\partial x}(q,x)f(q,x) \leq 0$ for all x since:

$$\begin{aligned}
\frac{\partial V}{\partial x}(q,x)f(q,x) &= \frac{d}{dt}V(q,x(t)) \\
&= \dot{x}^T P_i x + x^T P_i \dot{x} \\
&= x^T A_i^T P_i x + x^T P_i A_i x \\
&= x^T (A_i^T P_i + P_i A_i) x \\
&= -x^T I x \\
&= -\|x\|^2 \leq 0
\end{aligned}$$

It remains to test the non-increasing sequence condition. Notice that the level sets of $x^T P_i x$ are ellipses centered at the origin. Therefore each level set intersects the switching line $x_i = 0$ (for $i = 1, 2$) at exactly two points, $\hat{x}$ and $-\hat{x}$. Assume that $x(\tau_i) = \hat{x}$ and $q(\tau_i) = q_1$. The fact that $V(q_1, x(t))$ does not increase for $t \in [\tau_i, \tau_i']$ (where $q(t) = q_1$) implies that the next time a switching line is reached, $x(\tau_i') = \alpha(-\hat{x})$ for some $\alpha \in (0,1]$. Therefore, $\|x(\tau_{i+1})\| = \|x(\tau_i')\| \leq \|x(\tau_i)\|$. By a similar argument, $\|x(\tau_{i+2})\| = \|x(\tau_{i+1}')\| \leq \|x(\tau_{i+1})\|$. Therefore, $V(q(\tau_i), x(\tau_i)) \leq V(q(\tau_{i+2}), x(\tau_{i+2}))$.

3.1 Globally Quadratic Lyapunov Function

The material in this section and the next is drawn from the work of Mikael Johansson [4, 5], and also from [6, 7, 8].

Theorem 8 (Globally Quadratic Lyapunov Function) *Consider a hybrid automaton* $H = (S, \mathrm{Init}, f, \mathrm{Dom}, R)$ *with equilibrium* $x_e = 0$. *Assume that for all* i:

- $f(q_i, x) = A_i x, A_i \in \mathbb{R}^{n \times n}$
- $\mathrm{Init} \subseteq \mathrm{Dom}$
- *for all* $x \in \mathbb{R}^n$

$$|R(q_i,x)| = \begin{cases} 1 & \text{if } (q_i,x) \in \partial\mathrm{Dom} \\ 0 & \text{otherwise} \end{cases} \tag{25}$$

and $\{(q', x') \in R(q_i, x)\} \Rightarrow \{(q', x') \in \mathrm{Dom}, x' = x\}$. *Furthermore, assume that for all* $\chi \in \mathcal{E}_H^\infty$, $\tau_\infty(\chi) = \infty$. *Then, if there exists* $P = P^T > 0$ *such that*

$$A_i^T P + P A_i < 0, \forall i \tag{26}$$

$x_e = 0$ *is asymptotically stable.*

Proof: First note that there exists $\gamma > 0$ such that

$$A_i^T P + P A_i + \gamma I \leq 0, \forall i \tag{27}$$

Also, note that with the given assumptions there exists a unique, infinite, and non-Zeno execution $\chi = (\tau, q, x)$ for every $(q_0, x_0) \in$ Init. For all $i \geq 0$, the continuous evolution $x : \tau \to \mathbb{R}^n$ of such an execution satisfies the following time-varying linear differential equation:

$$\dot{x}(t) = \sum_i \mu_i(t) A_i x(t), t \in [\tau_i, \tau_i'] \tag{28}$$

where $\mu_i : \tau \to [0, 1]$ is a function such that for $t \in [\tau_i, \tau_i']$, $\sum_i \mu_i(t) = 1$. Letting $V(q, x) = x^T P x$, we have that for $t \in [\tau_i, \tau_i']$,

$$\begin{aligned} \dot{V}(q(t), x(t)) &= \sum_i [\mu_i(t) x(t)^T (A_i^T P + P A_i) x(t)] && (29) \\ &\leq -\gamma ||x(t)||^2 \sum_i \mu_i(t) && (30) \\ &= -\gamma ||x(t)||^2 && (31) \end{aligned}$$

Now, since $V(q, x) = x^T P x$, we have that

$$\lambda_{min} ||x||^2 \leq V(q, x) \leq \lambda_{max} ||x||^2 \tag{32}$$

where $0 < \lambda_{min} \leq \lambda_{max}$ are the smallest and largest eigenvalues of P respectively. It follows that

$$\dot{V}(q(t), x(t)) \leq -\frac{\gamma}{\lambda_{max}} V(q(t), x(t)), t \in [\tau_i, \tau_i'] \tag{33}$$

and hence

$$V(q(t), x(t)) \leq V(q(\tau_i), x(\tau_i)) e^{-\gamma(t-\tau_i)/\lambda_{max}}, t \in [\tau_i, \tau_i'] \tag{34}$$

Thus, from (32):

$$\lambda_{min} ||x(t)||^2 \leq \lambda_{max} ||x(\tau_i)||^2 e^{-\gamma(t-\tau_i)/\lambda_{max}}, t \in [\tau_i, \tau_i'] \tag{35}$$

Since the execution χ by assumption is non-Zeno, we have that $\tau_i \to \infty$ as $i \to \infty$. Hence, $||x(t)||$ goes to zero exponentially as $t \to \tau_\infty$, which implies that the equilibrium point $x_e = 0$ is asymptotically (actually exponentially) stable. ∎

Example 1: Consider the hybrid automaton H of Figure 8, with

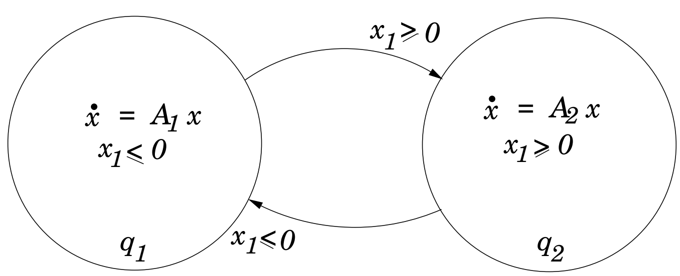

Figure 8: Example 1

$$A_1 = \begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix}, A_2 = \begin{bmatrix} -2 & 1 \\ -1 & -2 \end{bmatrix} \tag{36}$$

Since the eigenvalues of A_1 are $\lambda(A_1) = \{-1 \pm i\}$ and of A_2 are $\lambda(A_2) = \{-2 \pm i\}$, both $\dot{x} = A_1 x$ and $\dot{x} = A_2 x$ have an asymptotically stable focus. H satisfies the assumptions of the previous theorem; indeed, $A_1^T + A_1 < 0$ and $A_2^T + A_2 < 0$, so the inequalities in the theorem are satisfied for $P = I$. Hence, the origin is an asymptotically stable equilibrium point for H.

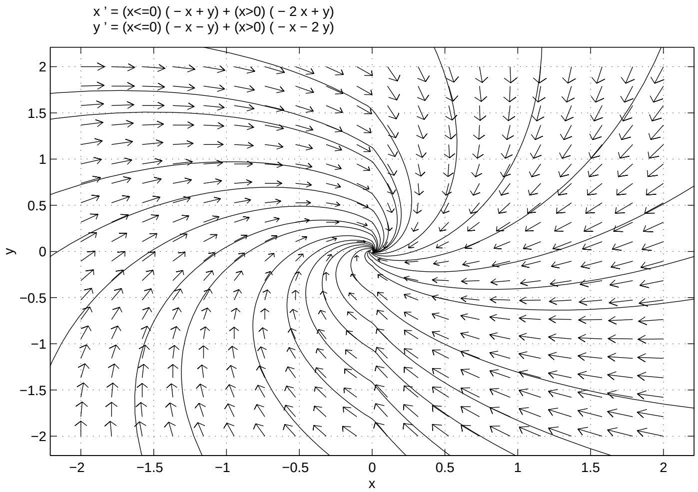

Figure 9: Example 1

Example 2 [5]: Consider the hybrid automaton from Figure 1 again, but let

$$A_1 = \begin{bmatrix} -5 & -4 \\ -1 & -2 \end{bmatrix}, A_2 = \begin{bmatrix} -2 & -4 \\ 20 & -2 \end{bmatrix} \tag{37}$$

The eigenvalues of A_1 are $\lambda(A_1) = \{-6, -1\}$ and of A_2 are $\lambda(A_2) = \{-2 \pm 4\sqrt{5}i\}$ so that $\dot{x} = A_1 x$ has an asymptotically stable node and $\dot{x} = A_2 x$ has an asymptotically stable focus. The evolution of the continuous state is shown in Figure 10 for four different initial states. The origin seems to be a stable equilibrium point – indeed, the Lyapunov function indicated by the dashed level sets proves asymptotic stability of the origin. Yet from the shape of its level sets, we see that the Lyapunov function is not globally quadratic, but piecewise quadratic, in the sense that it is quadratic in each discrete mode. (In fact, we can show for this example that it is not possible to find a quadratic Lyapunov function).

3.2 Piecewise Quadratic Lyapunov Function

If we assume that the hybrid automaton is restricted further so that the domains are given by polyhedra, then we can make some more general statements about the stability of the hybrid system:

$$\text{Dom} = \cup_i \{q_i\} \times \{x \in \mathbb{R}^n : E_{i1} x \geq 0, \ldots, E_{in} x \geq 0\} \tag{38}$$

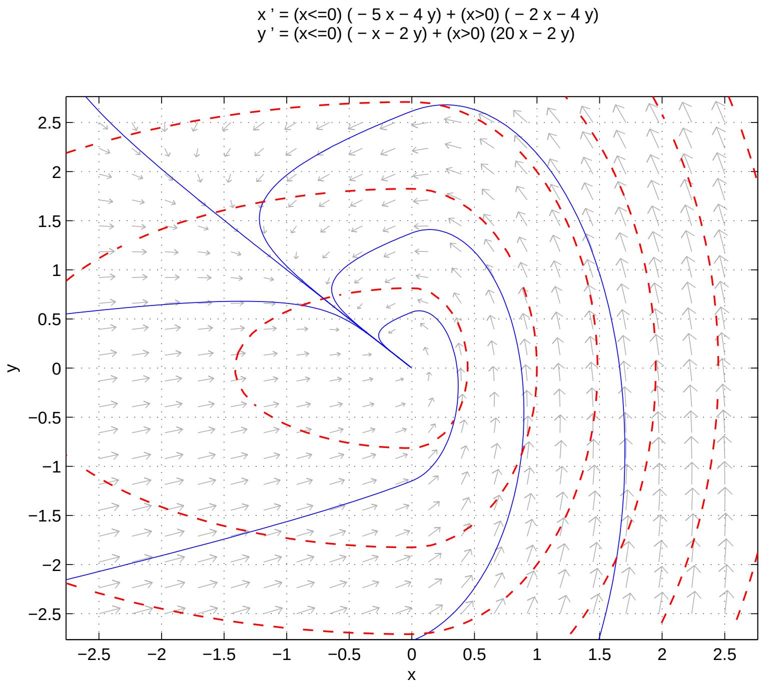

Figure 10: Continuous evolution for a hybrid automaton that does not have a globally quadratic Lyapunov function. Still, the origin is an asymptotically stable equilibrium point, which can be proved by using a Lyapunov function quadratic in each discrete state.

where

$$E_i = \begin{bmatrix} E_{i1} \\ \vdots \\ E_{in} \end{bmatrix} \in \mathbb{R}^{n \times n} \tag{39}$$

It then follows that $(q_i, 0) \in \text{Dom}$ for all i. Suppose that the reset relation R satisfies:

$$|R(q_i, x)| = \begin{cases} 1 & \text{if } (q_i, x) \in \partial\text{Dom} \\ 0 & \text{otherwise} \end{cases} \tag{40}$$

such that

$$(q_k, x') \in R(q_i, x) \Rightarrow F_k x = F_i x, q_k \neq q_i, x' = x \tag{41}$$

where $F_k, F_i \in \mathbb{R}^{m \times n}$ are given matrices (which hence define the boundaries of Dom). The LMI condition in Theorem 1 would require that

$$x^T(A_i^T P + PA_i)x < 0, \forall x \neq 0, (q_i, x) \in Q \times \mathbb{R}^n \tag{42}$$

It is however sufficient to require that

$$x^T(A_i^T P + PA_i)x < 0, \forall x \neq 0, (q_i, x) \in \text{Dom} \tag{43}$$

This can be done by specifying a matrix S_i such that $x^T S_i x \geq 0$ for all x with $(q_i, x) \in \text{Dom}$. Then,

$$A_i^T P + PA_i + S_i < 0 \tag{44}$$

still implies that

$$x^T(A_i^T P + PA_i)x < 0, \forall x \neq 0, (q_i, x) \in \text{Dom} \tag{45}$$

but $x^T(A_i^T P + PA_i)x < 0$ does not have to hold for $x \neq 0$ with $(q_k, x) \in \text{Inv}$ and $i \neq k$. The matrix S_i may be given as $S_i = E_i^T U_i E_i$, where E_i is given by the representation of H and $U_i = U_i^T \in \mathbb{R}^{n \times n}$ is chosen to have non-negative elements.

We can also let V depend on the discrete state, ie. $V(q_i, x) = x^T P_i x$ for $(q_i, x) \in \text{Inv}$. We choose $P_i = F_i^T M F_i$, where F_i is given by the representation of H, and $M = M^T \in \mathbb{R}^{n \times n}$ is to be chosen.

Theorem 9 (Piecewise Quadratic Lyapunov Function) $H = (S, \text{Init}, f, \text{Dom}, R)$ *with equilibrium* $x_e = 0$. *Assume that for all* i*:*

- $f(q_i, x) = A_i x, A_i \in \mathbb{R}^{n \times n}$
- $\text{Dom} = \cup_i \{q_i\} \times \{x \in \mathbb{R}^n : E_{i1}x \geq 0, \ldots, E_{in}x \geq 0\}$
- $\text{Init} \subseteq \text{Dom}$
- *for all* $x \in \mathbb{R}^n$

  $$|R(q_i, x)| = \begin{cases} 1 & \text{if } (q_i, x) \in \partial\text{Dom} \\ 0 & \text{otherwise} \end{cases} \tag{46}$$

  such that

  $$(q_k, x') \in R(q_i, x) \Rightarrow F_k x = F_i x, q_k \neq q_i, x' = x \tag{47}$$

  where $F_k, F_i \in \mathbb{R}^{n \times n}$*.*

Furthermore, assume that for all $\chi \in \mathcal{E}_H^\infty$*,* $\tau_\infty(\chi) = \infty$*. Then, if there exists* $U_i = U_i^T$*,* $W_i = W_i^T$*, and* $M = M^T$ *such that* $P_i = F_i^T M F_i$ *satisfies:*

$$A_i^T P_i + P_i A_i + E_i^T U_i E_i < 0 \tag{48}$$

$$P_i - E_i^T W_i E_i > 0 \tag{49}$$

where U_i, W_i *are non-negative, then* $x_e = 0$ *is asymptotically stable.*

Example 3: Consider the hybrid automaton of Figure 11 with

$$A_1 = A_3 = \begin{bmatrix} -0.1 & 1 \\ -5 & -0.1 \end{bmatrix}, A_2 = A_4 = \begin{bmatrix} -0.1 & 5 \\ -1 & -0.1 \end{bmatrix} \tag{50}$$

Here, we may choose

$$E_1 = -E_3 = \begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix}, E_2 = -E_4 = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \tag{51}$$

and

$$F_i = \begin{bmatrix} E_i \\ I \end{bmatrix} \forall i \in \{1, 2, 3, 4\} \tag{52}$$

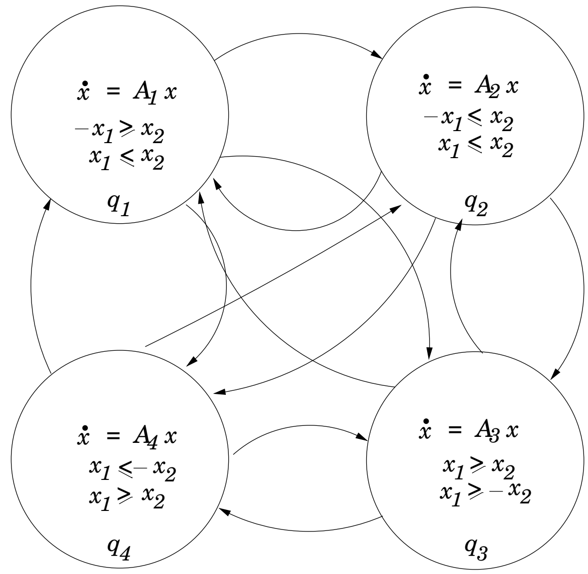

Figure 11: Example 3

The eigenvalues of A_i are $-1/10 \pm \sqrt{5}i$. The evolution of the continuous state is shown in Figure 12. We can use a Lyapunov function given by:

$$P_1 = P_3 = \begin{bmatrix} 5 & 0 \\ 0 & 1 \end{bmatrix}, P_2 = P_4 = \begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix} \tag{53}$$

to prove asymptotic stability of the hybrid automaton.

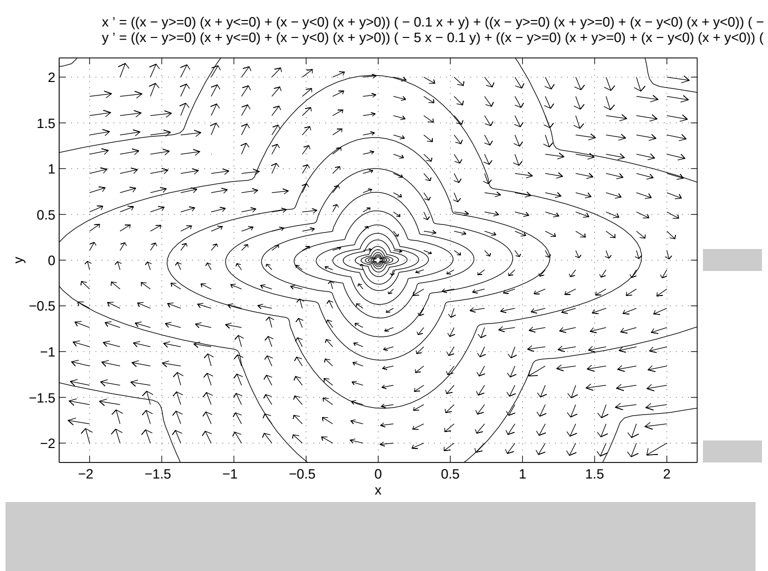

Figure 12: Example 3

3.3 Linear Matrix Inequalities

LMIs appear in many problems in systems and control theory (for example, see the reference [9]). For example, in the last Theorem we would like to solve

$$
\begin{aligned}
E_i^T M E_i &> 0 && (54)\\
A_i^T P_i + P_i A_i + E_i^T U_i E_i &< 0 && (55)\\
P_i - E_i^T W_i E_i &> 0 && (56)
\end{aligned}
$$

for the unknowns M, U_i, and W_i. This problem may be cast as an optimization problem, which turns out to be convex, so that it can be efficiently solved. In MATLAB, there is a toolbox called LMI Control Toolbox:

```
>> help lmilab
```

Try the demo:

```
>> help lmidem
```

and you'll see there is a graphical user interface to specify LMIs; you can enter this directly through:

```
lmiedit
```

After specifying an LMI, ie.:

$$
\begin{aligned}
P = P^T &\succ 0 && (57)\\
A^T P + PA &\prec 0 && (58)
\end{aligned}
$$

where A is a matrix that you've already entered in MATLAB's workspace in `lmisys`, a feasible solution P is found (if it exists) by running the commands:

```
[tmin, pfeas] = feasp(lmisys)
p = dec2mat(lmisys,pfeas,p)
```

The feasibility problem is solved as a convex optimization problem, which has a global minimum. This means that if the feasibility problem has a solution, the optimization algorithm will (at least theoretically) always find it.

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