HW#3 EECE 491M/571M Spr. 2008.

$\boxed{1}$ $\dot{x} = Ax + Bu$, $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

1. Case a) $u = -x_1 = [-1 \quad 0]\,x$

$$A_{cl_1} = A - BK_1 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix} [-1 \quad 0]$$

$$= \begin{bmatrix} 0 & 1 \\ -2 & 0 \end{bmatrix} \Rightarrow eig(A_1) =$$

$0 = \lambda^2 + 2 \Rightarrow \lambda_{1,2} = \pm j\sqrt{2}$. marginally stable, not asymptotically stable.

Case b) $u = 2x_1 = [2 \quad 0]\,x$

$$A_2 = A - BK_2 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix} [2 \quad 0] = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$0 = \lambda^2 - 1 \Rightarrow \lambda_{1,2} = \pm 1$. unstable.

2. $A_3 = \begin{bmatrix} -2 & 0 \\ 0 & 1 \end{bmatrix} \Rightarrow 0 = (\lambda+2)(\lambda-1) \Rightarrow \lambda_{1,2} = -2, +1$ unstable.

No GQLF, or quadratic CLF, exists since at least one mode is unstable. (both $A_2 + A_3$ have $\lambda_i > 0$).

3. $V(q, x) = x^T x$ ← a piecewise quadratic Lyap. function which happens to have the same form in each mode.

Note that $A_{eq} = \alpha A_1 + \beta A_2 + (1-\alpha-\beta) A_3$

$$= \begin{bmatrix} 0 & \alpha \\ -2\alpha & 0 \end{bmatrix} + \begin{bmatrix} 0 & \beta \\ \beta & 0 \end{bmatrix} + \begin{bmatrix} -2(1-\alpha-\beta) & 0 \\ 0 & (1-\alpha-\beta) \end{bmatrix}$$

$$A_{q_b} = \begin{bmatrix} -2(1-\alpha-\beta) & \alpha+\beta \\ -2\alpha+\beta & 1-\alpha-\beta \end{bmatrix}$$

$$|\lambda I - A_{eq}| = (\lambda + 2(1-\alpha-\beta))(\lambda - (1-\alpha-\beta)) - (-2\alpha+\beta)(\alpha+\beta)$$
$$= \lambda^2 + \lambda(1-\alpha-\beta) - 2(1-\alpha-\beta)^2 - (\alpha+\beta)(-2\alpha+\beta)$$

for stability, $\quad 1-\alpha-\beta > 0$

$$-2(1-\alpha-\beta)^2 - (\alpha+\beta)(-2\alpha+\beta) > 0$$
$$(2\alpha-\beta)(\alpha+\beta) > 2(1-\alpha-\beta)^2 > 0$$
$$\therefore \quad 2\alpha - \beta > 0.$$

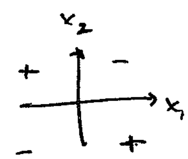
Choose $\alpha, \beta \in (0,1)$ s.t. $2\alpha > \beta$ $+ \alpha+\beta < 1$ to make A_{eq} have eigvals w/ negative real part $\Rightarrow$ there exists a stabilizing switching scheme.
E.g., $\alpha = 0.6$, $\beta = 0.2$.

To find stabilizing scheme (assuming $P = I$ solves the Lyap. eqn for A_{eq})*, find regions in each mode where

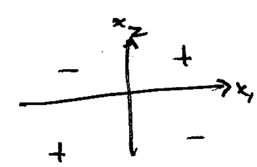
$$\frac{d}{dt} x^T x = x^T (A_q^T + A_q) x \le 0$$

Notice that $A_q^T + A_q$ is <u>indefinite</u> in all 3 modes.

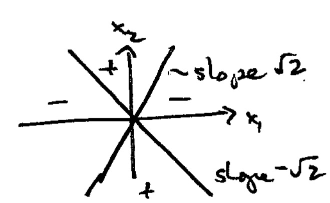
q_1: $x^T(A_1^T + A_1)x = x^T \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} x = -2x_1x_2 \le 0$ for quad. I + III.



q_2: $x^T(A_2^T + A_2)x = x^T \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix} x = +4x_1x_2 \le 0$ for quad. II + IV



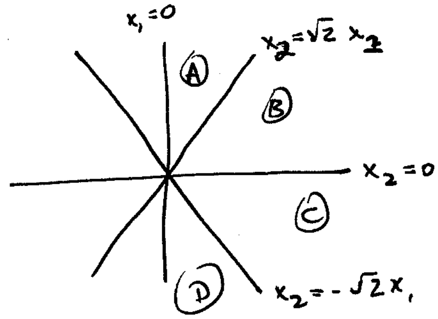
q_3: $x^T(A_3^T + A_3)x = x^T \begin{bmatrix} -4 & 0 \\ 0 & 2 \end{bmatrix} x = -4x_1^2 + 2x_2^2 \le 0$



Switching scheme is

$$\sigma(x) = \arg\min_{q} \; x^T(A_q^T + A_q)x$$

$$= \arg\min_{q} \{ -2x_1x_2, \; +4x_1x_2, \; -4x_1^2 + 2x_2^2 \}$$



In A: $x_1x_2 \geq 0$ and
$-4x_1^2 + 2x_2^2 \geq 0$

$\therefore \; -2x_1x_2 \leq 0 \leq 4x_1x_2$
and $\leq -4x_1^2 + 2x_2^2$

For A, $\sigma(x) = q_1$ for $x_1x_2 \geq 0, \; -4x_1^2 + 2x_2^2 \geq 0$

In B, $x_1x_2 \geq 0$ and
$-4x_1^2 + 2x_2^2 \leq 0$

$\therefore$ $-2x_1x_2 \leq 0$ and $-4x_1^2 + 2x_2^2 \leq 0$
$2x_1x_2 \geq 0$ $\quad 4x_1^2 - 2x_2^2 \geq 0$

$$4x_1^2 - 2x_2^2 \overset{?}{\geq} 2x_1x_2 \iff -4x_1^2 + 2x_2^2 \leq -2x_1x_2$$
$$\dot{V}(q_3, x) \leq \dot{V}(q_1, x)$$

$4x_1^2 - 2x_1x_2 - 2x_2^2 \geq 0$

$2x_1^2 - x_1x_2 - x_2^2 \geq 0$

$(2x_1 + x_2)(x_1 - x_2) \geq 0$ $\qquad$ for $x_1 \geq x_2$.

For B, $\sigma(x) = \begin{cases} q_3 & \text{for } x_1 \geq x_2, \; x_1x_2 \geq 0, \; -4x_1^2 + 2x_2^2 \leq 0 \\ q_1 & x_1 < x_2, \; x_1x_2 \geq 0, \; -4x_1^2 + 2x_2^2 \leq 0 \end{cases}$

In C, $4x_1x_2 \leq 0$ and $-4x_1^2 + 2x_2^2 \leq 0$

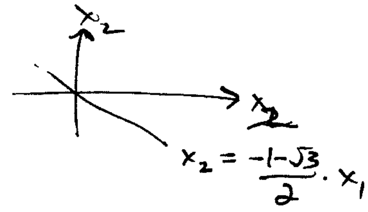
$$-4x_1^2 + 2x_2^2 \overset{?}{\leq} 4x_1x_2 \iff \dot{V}(q_3, x) \leq \dot{V}(q_2, x)$$

$0 \leq 4x_1^2 + 4x_1x_2 - 2x_2^2$

$0 \leq 2x_1^2 + 2x_1x_2 - x_2^2$

Solns to this occur at $\dfrac{x_2}{x_1} = \dfrac{-2 \pm \sqrt{2^2 - 4(2)(-1)}}{2 \cdot 2} = \dfrac{-1 \pm \sqrt{3}}{2}$

$\therefore$ Since in Quad IV $x_1 > 0$ + $x_2 < 0$,



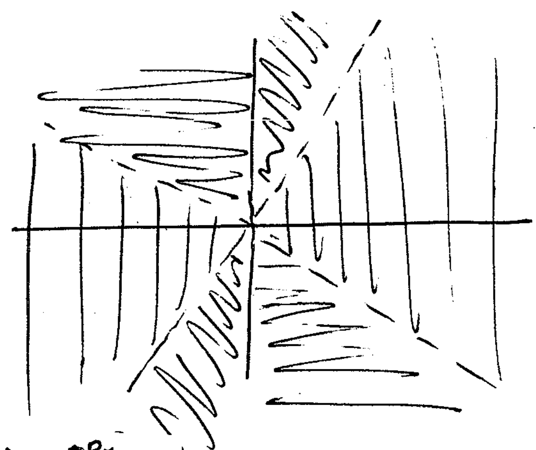
$$\frac{x_2}{x_1} = \frac{-1-\sqrt{3}}{2} \Rightarrow x_2 = \frac{-1-\sqrt{3}}{2} \cdot x_1$$

For (C), $\sigma(x) = \begin{cases} q_3 & \text{for } x_2 \geq \frac{-1-\sqrt{3}}{2} x_1, \\ q_2 & \text{o/w.} \end{cases}$

In (D), $4x_1x_2 \leq 0$ and $-4x_1^2 + 2x_2^2 \geq 0$

For (D), $\sigma(x) = q_2$.

Therefore by symmetry:



$\equiv$: q_1

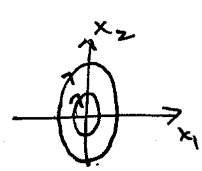
$|||$: q_3

$\equiv$: q_2

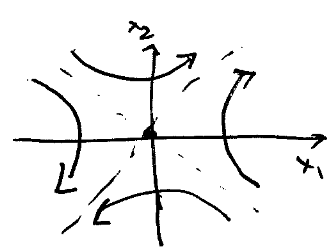
$$\sigma(x) = \begin{cases} q_1 & \text{for } \begin{cases} x_1 \geq 0, x_2 \geq 0, x_2 \geq \sqrt{2}x_1 & \text{OR} \\ x_1 \leq 0, x_2 \leq 0, x_2 \leq \sqrt{2}x_1 \end{cases} \\ q_3 & \text{for } \begin{cases} x_1 \geq 0, & x_2 \leq \sqrt{2}x_1, x_2 \geq \frac{-1-\sqrt{3}}{2} \cdot x_1 & \text{OR} \\ x_1 \leq 0, & x_2 \geq \sqrt{2}x_1, x_2 \leq \frac{-1-\sqrt{3}}{2} \cdot x_1 \end{cases} \\ q_2 & \text{for } \begin{cases} x_1 \geq 0, & x_2 \leq 0, x_2 \leq \frac{-1-\sqrt{3}}{2} x_1 & \text{OR} \\ x_1 \leq 0, & x_2 \geq 0, x_2 \geq \frac{-1-\sqrt{3}}{2} x_1 \end{cases} \end{cases}$$

4. Sketch of trajectories.

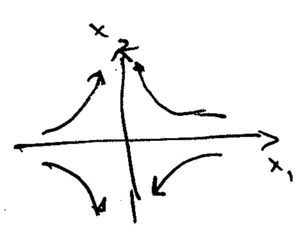
Since: q_1:



q_2:

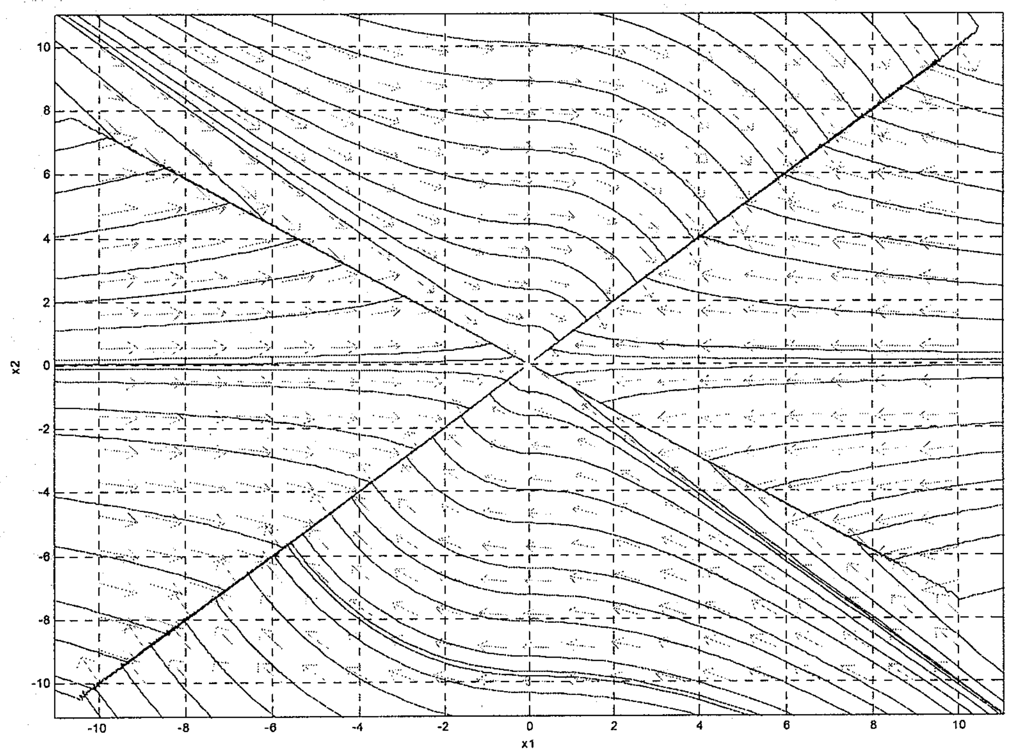


q_3:

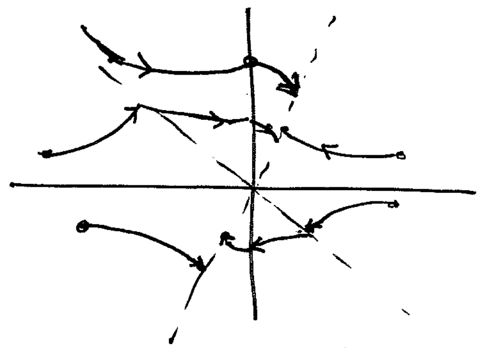


From E. Miehling

4. Using the boundaries derived from the previous question, the phase portrait below was obtained. Exact equations are attached to this page (hand-written).



Very good!



Note this looks like it
might be zero.

(*) Some people correctly noticed that $P = I$ does not actually fulfill the Lyapunov equation for A_{eq}.
A ~~alternative~~ proper, correct solution showed complete the above analysis for a P which does fulfill the eqn

$$A_{eq}^T P + P A_{eq} = -Q_{eq}$$

for some positive definite Q_{eq}.

$\boxed{2}$ 1. $\lambda^1_{1,2} = -a_1, -c_1$ so A_1, A_2 are stable.

$\lambda^2_{1,2} = -a_2, -c_2$

Transform both dynamics by $T = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = T^{-1}$ to make $\tilde{A}_1 = T^{-1} A_1 T$, $\tilde{A}_2 = T^{-1} A_2 T$ upper triangular.

$\therefore$ CLF exists.

2. $\lambda^1_{1,\dots,n} = d_1, \dots, d_n < 0$

$\lambda^2_{1,\dots,n} = Yd_1, \dots, Yd_n < 0$

$\lambda^3_{1,\dots,n} = 1, \dots, 1 > 0 \longrightarrow$ one mode is unstable.

$\therefore$ No CLF exists.

3. $\lambda^1_{1,2,3} = -1,-1,-1$ and $A_1 A_2$ are lower-triangular.
$\lambda^2_{1,2,3} = -1,-1,-1$

Also note $A_1 A_2 = A_2 A_1$

$\therefore$ CLF exists.

4.

From Ehsan Azadi Yazdi

Part 4

Consider the following switched linear system,

$$A_i = QR_i \qquad ,i \in \{1,\ldots,m\} \tag{14}$$

with Q an orthogonal matrix, R_i an upper-triangular matrix, and A_i Hurwitz.

In QR decomposition, Q matrix can be determined by Gram-Schmidth process. Based on the QR decomposition process we can easily see that each column of Q matrix one of the generalized eigenvectors of A (ref.: http://tutorial.math.lamar.edu/Classes/LinAlg/QRDecomposition.aspx).

To prove the stability of this hybrid system with arbitrary switching scheme, we are looking for a common nonsingular matrix T such that the matrices $\tilde{A}_i = T^{-1}A_iT$ are upper-triangular.

Since Q is the modal matrix of A (i.e., resulting from Gram-Schmidth process) we can write the following equation.

$$Q^{-1}A_iQ = \begin{bmatrix} J_1 & 0 & \cdots & 0 \\ 0 & J_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & J_m \end{bmatrix} \tag{15}$$

Where $J_1,\ldots,J_m$ are Jordan blocks. Since Jordan blocks are upper triangular, the above matrix will be upper triangular for all i. Hence we can use Q as the T matrix in the stability theory.

$$\tilde{A}_i = Q^{-1}A_iQ \tag{16}$$

Since A_is are Hurwitz, this class of hybrid systems is asymptotically stable for all switching schemes. Therefore a common Lyapunov function exists for this system.