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Lecture 5

Stability of Continuous Systems

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Stability lectures

- Linear system stability
  - Eigenvalues of A
  - Linear quadratic Lyapunov functions
  - Ellipses / LMIs
- Nonlinear system stability
  - Lyapunov's indirect method
  - Lyapunov's direct method
- Hybrid system stability
  - Definition of equilibrium
  - Multiple Lyapunov functions
  - Common Lyapunov function
  - Piecewise quadratic Lyapunov functions

Stability of Linear Systems

Linear System Asymptotic Stability Theorem:

- The autonomous system

  $dx/dt = A x,$

  $x(0) = x_0$

  is asymptotically stable **if and only if** the eigenvalues of A have strictly negative real part:

  $\lambda_i(A) < 0.$

- The trajectories of the system will follow

  $x(t) = \exp(At)\, x_0$

  which converges exponentially to 0 as x -> ∞.

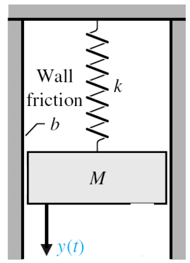
Stability of Linear Systems

- LTI systems can be represented in a variety of coordinate systems, yet all representations share the same stability properties.
- Exercise: Consider the invertible transformation

  $z = T^{-1}x.$
  - What are the eigenvalues of the transformed system matrix? (Use Cayley-Hamilton theorem.)

Linear system stability

- Example: Spring-mass-damper system
  - Spring constant k
  - Damping coefficient b
  - Mass m



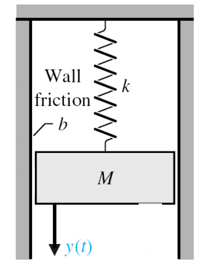
$$m\ddot{y} = -ky - b\dot{y}$$

$$x_1(t) = y(t), x_2(t) = \frac{dy(t)}{dt}$$

$$\begin{bmatrix} \frac{dx_1(t)}{dt} \\ \frac{dx_2(t)}{dt} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{M} & -\frac{b}{M} \end{bmatrix} x(t)$$

Linear system stability

- Example: Spring-mass-damper system
  - Eigenvalues occur where
    $0 = \lambda^2 + \lambda b/m + k/m$
  - And have negative real part for
    $b > 0, k > 0$



Phase-plane analysis

- For 2-D linear systems, the **phase plane plot** is a plot of trajectories in (x_1, x_2) space.
- Stable trajectories will tend towards the origin and can be classified according to the types of eigenvalues:
  - Both negative real numbers (**stable node**)
  - Complex conjugate pair with negative real part (**stable focus**)
- Unstable trajectories tend towards infinity and can be classified according to the types of eigenvalues:
  - Both positive real numbers (**unstable node**)
  - Complex conjugate pair with positive real part (**unstable focus**)
  - Positive and negative real numbers (**saddle**)
- What happens when one eigenvalue has 0 real part?
- What happens when both eigenvalues have 0 real part?
- What happens when eigenvalues are repeated?

Lyapunov stability for LTI sys.

- For the dynamical system
  $$\dot{x} = Ax, \quad x(0) = x_0$$
- consider the **quadratic Lyapunov function**
  $$V(x) = x^T P x, \quad P = P^T$$
- whose time-derivative
  $$\begin{aligned} \dot{V}(x) &= x^T P \dot{x} + \dot{x}^T P x \\ &= x^T P A x + (Ax)^T P x \\ &= x^T (PA + A^T P) x \end{aligned}$$
- can be written as
  $$\dot{V} = -x^T Q x, \quad Q \triangleq -(PA + A^T P) = Q^T$$

Lyapunov stability for LTI sys.

Theorem: Lyapunov stability for linear systems

- The system $\dot{x} = Ax, x(0) = x_0$
  is asymptotically stable about x=0 **if and only if** for any positive definite Q, there exists a positive definite P such that
  $$A^T P + PA = -Q$$
  This is known as the Lyapunov equation.

- Further, $$P = \int_0^\infty e^{A^T t} Q e^{At} dt$$
  uniquely solves the Lyapunov equation.

Positive definite matrices

- A matrix P is **positive definite** iff
  $$x^T P x > 0, \quad x \neq 0$$
  $$x^T P x = 0, \quad x = 0$$
- A matrix P is **positive semi-definite** iff
  $$x^T P x \geq 0$$
- A matrix P is **negative definite** iff
  $$x^T P x < 0, \quad x \neq 0$$
  $$x^T P x = 0, \quad x = 0$$
- A matrix P is **negative semi-definite** iff
  $$x^T P x \leq 0$$

Positive definite matrices

- Any real, symmetric positive definite matrix $P=P^T > 0$ has real eigenvalues $\lambda_i > 0$
- Note that P can be diagonalized by an orthonormal basis
  $$P = \tilde{Q}\Lambda\tilde{Q}^T, \quad \tilde{Q}^T\tilde{Q} = I, \Lambda = \mathrm{diag}(\lambda_1, \cdots, \lambda_n)$$
  where the columns of $\tilde{Q}$ are normalized, orthogonal eigenvectors (even if there are repeated eigenvalues)
- Values of $V(x)=x^T Px$ are bounded by
  $$\lambda_{\min}(P)x^T x \leq x^T P x \leq \lambda_{\max}(P)x^T x$$

Positive definite matrices

- Evaluate the following matrices for positive definiteness:

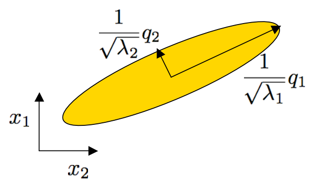
$$P = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} \qquad P = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

$$P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad P = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$$

$$P = \begin{bmatrix} 3 & 0 \\ 0 & 0 \end{bmatrix} \qquad P = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}$$

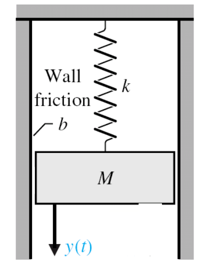
Positive definite matrices

- Interpretation: The function
  $$V(x) = x^T P x \le c$$
  represents an ellipse in R^n with axes along the eigenvectors of P
- Consider the 2D case:
  For c=1, with $\lambda_2 \ge \lambda_1 > 0$
  $$\frac{1}{\sqrt{\lambda_2}} \le \frac{1}{\sqrt{\lambda_1}}$$



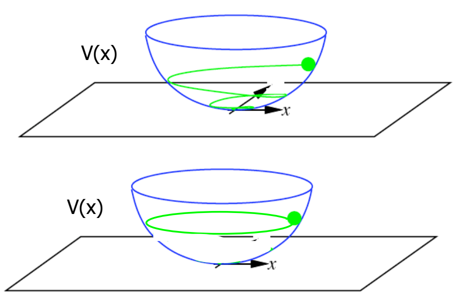
Example: SMD system

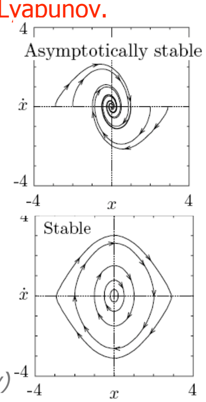
- Consider the standard autonomous spring-mass-damper system.
  $$m\ddot{x} = -kx - b\dot{x}$$
- Assume m=1, b=2, k=1.
- Choose Q = I.
- By computing a P that is positive definite, we prove asymptotic stability.
  $$A^T P + PA = -I$$



Lyapunov stability

- What if asymptotic stability is not possible?
- If V always decreases, then the system is asymptotically stable
- If V decreases or maintains a constant value as time increases, then the system is stable in the sense of Lyapunov.





(From Murray, Li, Sastry)

Lyapunov stability

- Definition:
  The equilibrium point x* of
  dx/dt = f(x), x(0) = x_0
  is stable in the sense of Lyapunov if for all $\varepsilon > 0$ there exists a $\delta > 0$ such that
  $$||x_0|| < \delta \Rightarrow ||x(t)|| < \epsilon, \forall t \ge 0$$

- Definition:
  The equilibrium point x* is asymptotically stable if it is stable and δ can be chosen such that
  $$||x_0|| < \delta \Rightarrow \lim_{t \to \infty} ||x(t)|| = 0$$

Lyapunov stability for LTI sys.

Linear quadratic Lyapunov theorems

- If P > 0, Q > 0, then system is asymptotically stable
- If P > 0, Q ≥ 0, then system is stable in the sense of Lyapunov
- If P > 0, Q ≥ 0, and (Q, A) observable, then system is asymptotically stable
- If P > 0, Q ≥ 0, the sublevel sets of { x | $x^T P x \le a$ } are invariant and are ellipsoids
- If P ≥ 0, Q ≥ 0, then the system is **not** stable.

Lyapunov stability

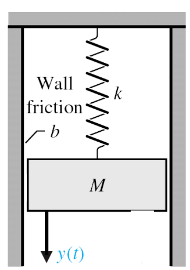
Converse linear quadratic Lyapunov theorems

- If A is asymptotically stable, then there exists P > 0, Q > 0 that satisfy the Lyapunov equation
- If A is stable and Q ≥ 0, then P ≥ 0
- If A is stable, Q ≥ 0, and (Q, A) is observable, then P > 0

Example: SMD system

- Consider the spring-mass-damper system again.
- Other Lyapunov functions are possible.
  - Case 1: $V(x) = \frac{1}{2}mv^2 + \frac{1}{2}kx^2$

    Proves stability in the sense of Lyapunov, since $\dot{V}(x) = \frac{\partial V(x)}{\partial x} f(x) = -bv^2 \le 0 \text{ for all } x$
  - Case 2: $V(x) = \frac{1}{2}mv^2 + \frac{1}{2}kx^2 + \frac{1}{2}bxv$
    Proves asymptotic stability since dV/dt < 0
- Now evaluate each of these functions in terms of the linear quadratic Lyapunov and converse linear quadratic Lyapunov theorems



Example: SMD system

- Case 1:
  - P > 0, Q ≥ 0 --> therefore the system is stable.
  - But is (Q, A) observable?
- Case 2:
  - P > 0, Q > 0 --> therefore the system is globally asymptotically stable.
- By the converse Lyapunov theorem, we know that since eig(A) < 0 a quadratic Lyapunov function must exist.

Example 2

- Consider the linear system with
  - A = [-1 10; -100 1]
- Does a quadratic Lyapunov function that satisfies the Lyapunov equation exist?

Solving the Lyapunov equation

Integral solution to the Lyapunov equation

- If dx/dt = Ax is asymptotically stable and $Q = Q^T > 0$, **OR** if dx/dt = Ax is stable in the sense of Lyapunov & $Q = Q^T \geq 0$,

$$P = \int_0^\infty e^{A^T t} Q e^{At} dt$$

is the unique solution to the Lyapunov equation

$$A^T P + PA + Q = 0$$

- If you know Q, you can also use `P = lyap(A',Q)` in Matlab

Solving the Lyapunov equation

- To show stability, we want to find a positive definite matrix P such that

$$A^T P + PA < 0$$
$$P > 0$$

- The variable in these matrix equations is the matrix P
- This is known as a **Linear Matrix Inequality**
- Efficient tools have been developed to quickly solve LMIs by posing them as convex optimization problems.
  - If the problem has a solution, the algorithm will find it.
  - If the problem does not have a solution, the algorithm will return a certificate which indicates as such.
- The Matlab LMI Control Toolbox can solve this in $O(n^3)$.

Solving the Lyapunov equation

- GUI to specify LMIs
  `>> lmiedit`
- The system of LMIs is encoded in `lmisys`
- The optimization to find a feasible solution to the LMI is called through

```
>> [tmin, pfeas]=feasp(lmisys)
>> p = dec2mat(lmisys, pfeas, p)
```

- More help can be found at
  `>> help lmidem`
- Or through the demo
  `>> help lmidem`
- More on this later…

Summary: Linear Sys. Stability

- Quadratic Lyapunov functions for linear systems
- Positive definite matrix properties
- Linear quadratic Lyapunov stability theorem for linear systems
- Necessary and sufficient conditions for stability (special case for linear systems)
  - Converse theorems
  - Asymptotic stability vs. stability in the sense of Lyapunov
- Tools to solve the Lyapunov equation and LMIs

Extensions to nonlinear sys.

- Nonlinear systems have significant differences that complicate stability analysis.
- As opposed to linear systems, nonlinear systems can have **multiple equilibria**.
- As opposed to linear systems, nonlinear system **stability is often only a local** result (e.g., valid within some neighborhood of the equilibrium point).
- As opposed to linear systems, nonlinear systems **rarely have closed-form solutions** (e.g., there is no $x(t) = e^{At} x(0)$).
- In addition to the behavior around equilibria that arose in linear systems, nonlinear systems may exhibit **orbits, limit cycles, bifurcations**, and other phenomena.

Nonlinear System Stability

Theorem: Lyapunov's indirect method

- Let x*=0 be the equilibrium of the differentiable function
  $dx/dt = f(x),\ x(0) = x_0$
  and let $D \subset \mathbb{R}^n$ be a set containing x*. Let
  $$A = \frac{\partial f}{\partial x}|_{x=x_0}$$
  such that the linearized system is $\dot{z} = Az,\ z = x - x_0$
- Then
  - x* is asymptotically stable if $\mathrm{Re}(\lambda_i)<0$ for all eigenvalues of A.
  - x* is unstable if $\mathrm{Re}(\lambda_i)>0$ for at least one eigenvalue of A.

Nonlinear System Stability

- Lyapunov's "second method" or "direct method"

- **Theorem**:
  Let x*=0 be the equilibrium of
  $dx/dt = f(x),\ x(0) = x_0$
  and let $D \subset \mathbb{R}^n$ be a set containing x*. If V: D->R is a continuously differentiable function such that
  $$\begin{aligned} V(0) &= 0 \\ V(x) &> 0, \forall x \in D\backslash\{0\} \\ \dot{V}(x) &\leq 0, \forall x \in D \end{aligned}$$
  then x* is stable. Further more, if x*=0 is stable and
  $$\dot{V}(x) < 0, \forall x \in D\backslash\{0\}$$
  then x* is asymptotically stable.

Lyapunov stability

- Lyapunov function is sufficient condition for stability
- Evaluating eigenvalues is necessary and sufficient for stability
- Allows trajectories which do not converge to the origin to be “stable”.
- If the system is stable, then there exists a Lyapunov function.
- If a Lyapunov function cannot be found, nothing is known about the stability of the system.
- For general nonlinear systems, these functions can be hard to find.
- Recent computational tools in LMIs and polynomial functions can provide numerical computations of Lyapunov functions.

Summary: Nonlinear sys.

- Stability in the sense of Lyapunov
  - Indirect method:
    - If the linearization is asymptotically stable, then the nonlinear system is locally asymptotically stable.
    - If the linearization is unstable, then the nonlinear system is locally unstable.
    - In general, **no conclusions are possible** regarding the nonlinear system if the eigenvalues have 0 real part. (Some exceptions for 2D systems -- Hartman-Grobman theorem)
  - Direct method:
    - If you can find a Lyapunov function, then you know the system is locally stable in the sense of Lyapunov.
    - Sufficient condition for stability

Next couple of weeks

- Introduction to hybrid stability
  - Hybrid equilibrium
  - Hybrid stability
- Multiple Lyapunov functions
  - (Most general stability theory)
- Global quadratic Lyapunov functions
  - (specific to hybrid systems with linear dynamics and arbitrary switching)
- Piecewise quadratic Lyapunov functions
  - (hybrid systems with linear or affine dynamics and state-based switching)