EECE 571M/491M, Spring 2008

Lecture 8

Piecewise Linear Quadratic Lyapunov Theory

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Tomlin LN 6, Johansson and Rantzer (1998), DeCarlo et al (2000)

Today's lecture

- Review
  - Hybrid equilibrium
  - Hybrid stability
  - Multiple Lyapunov functions
- Global Linear Quadratic Lyapunov function
  - (Results from Johansson and Rantzer 1998)
- Piecewise Linear Quadratic Lyapunov functions
  - (Results from Johansson and Rantzer 1998)

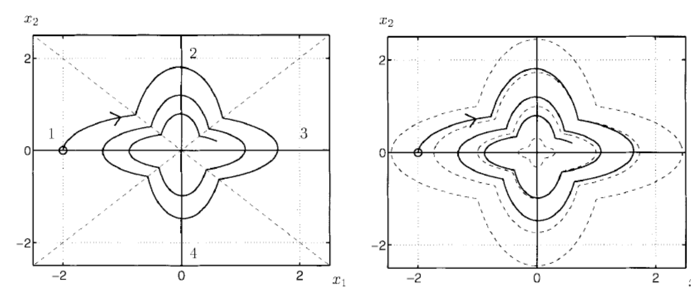
Globally Quad. Lyapunov Fcn

- Consider the case when
  - State-space is partitioned into disjoint regions
  - Reset map is the identity
  - Dynamics are linear and asymptotically stable in each mode
  - Only one transition is possible from each mode
- The same linear quadratic Lyapunov function is used in each mode (hence 'global')
  - $V_i(x) = x^T P x$ for all modes

  but $-Q_i = A_i^T P + P A_i$ may be different each mode
- Multiple Lyapunov Function theorem
  - is satisfied for $P > 0$, $Q_i > 0$.
  - can be simplified into the Globally Quadratic Lyapunov Function theorem.

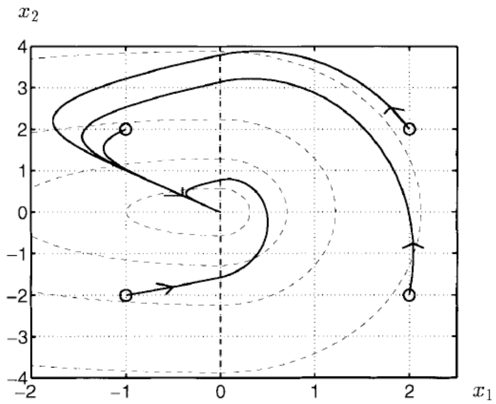
GQLF Example #2

- Try A1 = [-0.1 1; -5 -0.1] =A3 and A2 = [-0.1 5; -1 -0.1] = A4 (stable, but no quadratic Lyapunov function)



GQLF Example #3

- Try A1 = [-5 -4; -1 -2] and A2 = [-2 -4; 20 -4] (stable, but no quadratic Lyapunov function)



Relaxing the Global QLF

- Switched systems (continuity of the state)
- Only need to have positive definiteness within sectors (not R^n)
- Computationally done through "S-procedure"
- Still solve a set of LMIs

Piecewise Quad. Lyap. fun

- When no common Lyapunov function exists
- Different Lyapunov-like functions in each mode
- Disjoint partition of the state-space
  - Described by intersections of hyperplanes
  - Domain is set of convex polyhedra
- Linear dynamics in each mode

- Goal: Find P_i such that
  - $P_i > 0$ for x in X_i
  - $A^T P_i + P_i A < 0$ for x in X_i
  - and V(p,x) maintains continuity across modes

Piecewise Quad. Lyap. fun

Theorem 9 (Piecewise Quadratic Lyapunov Function) *$H = (S, \mathrm{Init}, f, \mathrm{Dom}, R)$ with equilibrium $x_e = 0$. Assume that for all i:*

- $f(q_i, x) = A_i x$, $A_i \in \mathbb{R}^{n\times n}$
- $\mathrm{Dom} = \cup_i \{q_i\} \times \{x \in \mathbb{R}^n : E_{i1}x \geq 0, \ldots, E_{in}x \geq 0\}$
- $\mathrm{Init} \subseteq \mathrm{Dom}$
- *for all* $x \in \mathbb{R}^n$
$$|R(q_i, x)| = \begin{cases} 1 & \text{if } (q_i, x) \in \partial\mathrm{Dom} \\ 0 & \text{otherwise} \end{cases} \qquad (46)$$
such that
$$(q_k, x') \in R(q_i, x) \Rightarrow F_k x = F_i x, q_k \neq q_i, x' = x \qquad (47)$$
where $F_k, F_i \in \mathbb{R}^{n\times n}$.

Furthermore, assume that for all $\chi \in \mathcal{E}_H^\infty$, $\tau_\infty(\chi) = \infty$. *Then, if there exists* $U_i = U_i^T$, $W_i = W_i^T$, *and* $M = M^T$ *such that* $P_i = F_i^T M F_i$ *satisfies:*

$$A_i^T P_i + P_i A_i + E_i^T U_i E_i < 0 \qquad (48)$$
$$P_i - E_i^T W_i E_i > 0 \qquad (49)$$

where U_i, W_i *are non-negative, then* $x_e = 0$ *is asymptotically stable.*

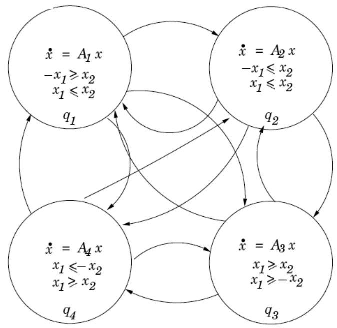
Piecewise Quad. Lyap. fun

$$A_1 = A_3 = \begin{bmatrix} -0.1 & 1 \\ -5 & -0.1 \end{bmatrix}, A_2 = A_4 = \begin{bmatrix} -0.1 & 5 \\ -1 & -0.1 \end{bmatrix}$$

$$E_1 = -E_3 = \begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix}, E_2 = -E_4 = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$$

$$F_i = \begin{bmatrix} E_i \\ I \end{bmatrix} \forall i \in \{1, 2, 3, 4\}$$

$$P_1 = P_3 = \begin{bmatrix} 5 & 0 \\ 0 & 1 \end{bmatrix}, P_2 = P_4 = \begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix}$$



Summary

- Global Quadratic Lyapunov functions
  - Easiest to start with
  - Irrespective of switching strategy
  - Solved by LMIs
- Piecewise Quadratic Lyapunov functions
  - When no global quadratic Lyapunov function exists
  - Solved by LMIs
  - Switched linear dynamics