EECE 571M/491M, Spring 2008
Lecture 12

Piecewise Quadratic Optimal Control

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Rantzer and Johansson (2000),
Lazar, Bemporad et al. (2006)

Today's lecture

- Background
  - Optimal control for continuous linear systems (VERY cursory!)
- Literature survey and related topics
- Bounds on optimal costs
  - Johansson and Rantzer
  - Lincoln and Rantzer
  - Hedlund and Rantzer
- Stabilizing MPC
  - Lazar, Heemels, Weiland, and Bemporad
  - Bemporad, Borrelli, Morari

Linear Quadratic Optimal Ctrl

- **Note**: This is a very cursory look at a deep topic. Focus here is on summarizing most relevant results.
- For more information
  - R. Stengel, 1994.
  - Bertsekas, 1995.
- Optimal controls course, P. Loewen, Math, UBC

Current research

- S. Hedlund and A. Rantzer, CDC 1999.
  - Nonlinear; fixed initial and final modes and states; unknown sequences
- M. Johansson and A. Rantzer, TAC 2000.
  - Piecewise linear quadratic systems
- B. Lincoln and A. Rantzer, TAC 2002.
  - Relaxed dynamic programming
- A. Bemporad, F. Borelli, M. Morari, Automatica 2002.
  - Discrete-time mixed-logical dynamical systems
- M. Lazar, P. Heemels, S. Weiland, A. Bemporad, TAC 2006.
  - (Stabilizing) model predictive control for switched systems
- X. Xu and P. Antsaklis, TAC 2004.
  - Order of switching sequence known
  - Timing and continuous control input unknown
- M. Boccadoro, Y. Wardi, M. Egerstedt, E. Verriest, 2005.
  - Parameterization of switching surfaces
- Many others…

Piecewise LQR

- A variety of formulations of optimality problems exist.
- Switched systems:
  - Optimal switching sequence (modes and switching times)
  - Optimal continuous control within each mode
- Brute-force computations lead to combinatorial explosions in the exploration of all switching alternatives
- Continuous-time vs. discrete-time
- Solution construction vs. bounds

Piecewise LQR

- Consider PWA systems with the following minimization problem

$$\begin{aligned} &\text{Minimize} \quad \int_0^\infty l(x,u)\,dt \\ &\text{subject to} \quad \begin{cases} \dot{x}(t) = f(x(t), u(t)) \\ x(0) = x_0. \end{cases} \end{aligned}$$

- Goal: Determine lower bound on optimal cost
- "Optimal" control law can be synthesized based on this result
- Further work must be done to find an upper bound

From Rantzer and Johansson, TAC 2000.

Piecewise LQR

- Theorem: Lower bound on optimal cost

Theorem 2 (Lower Bound on Optimal Cost): Assume existence of symmetric matrices T and U_i, such that U_i have nonnegative entries, while $P_i = F_i' T F_i$ and $\overline{P}_i = \overline{F}_i' T \overline{F}_i$ satisfy

$$\begin{aligned} 0 &< \begin{bmatrix} P_i A_i + A_i' P_i + Q_i - E_i' U_i E_i & P_i B_i \\ B_i' P_i & R_i \end{bmatrix} \qquad i \in I_0 \\ 0 &< \begin{bmatrix} \overline{P}_i \overline{A}_i + \overline{A}_i' \overline{P}_i + \overline{Q}_i - \overline{E}_i' U_i \overline{E}_i & \overline{P}_i \overline{B}_i \\ \overline{B}_i' \overline{P}_i & R_i \end{bmatrix} \qquad i \in I_1. \end{aligned}$$

Then, every continuous piecewise C^1 trajectory $x(t) \in \cup_{i \in I} X_i$ of (1) with $x(\infty) = 0$, $x(0) = x_0 \in X_{i_0}$ satisfies

$$J(x_0, u) \geq \sup_{T, U_i} \overline{x}_0' \overline{P}_{i_0} \overline{x}_0.$$

From Rantzer and Johansson, TAC 2000.

Piecewise LQR

- This bound is based on a non-optimal value function V(x)
- A control law which satisfies

$$\min_u \left(\frac{\partial V}{\partial x} f(x,u) + l(x,u) \right)$$

will achieve the sub-optimal cost $J(x_0, u)$.

$$\begin{aligned} L_i &= -R_i^{-1} B_i' P_i \\ \overline{L}_i &= -R_i^{-1} \overline{B}_i' \overline{P}_i \end{aligned}$$

$$u(t) = \overline{L}_i \overline{x}, \qquad x \in X_i$$

From Rantzer and Johansson, TAC 2000.

Discrete-time piecewise LQR

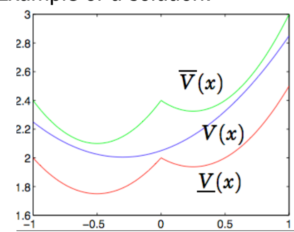
- Alternative way to solve for optimal cost: **Relaxed dynamic programming** (Lincoln, Rantzer, 2003)
- Instead of solving dynamic program exactly, solve the inequalities

$$\min_u \left\{ V\left(f(x,u)\right) + \underline{l}(x,u) \right\} \le V(x) \le \min_u \left\{ V\left(f(x,u)\right) + \overline{l}(x,u) \right\}$$

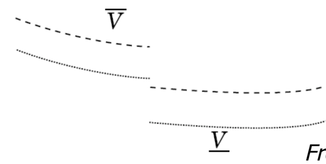
- Discrete-time formulation

Discrete-time piecewise LQR

- Example of a solution:



- Example of a problem:



From Lincoln and Rantzer, CDC 2003

Optimization through MPC

- Model Predictive Control
- MPT (Multi-Parametric Toolbox) computes
  - Terminal state constraints to ensure stability
  - Optimal, stabilizing control law
  - Optimal cost
- Equivalence to mixed logical dynamical systems
- Mixed integer quadratic program (MIQP)

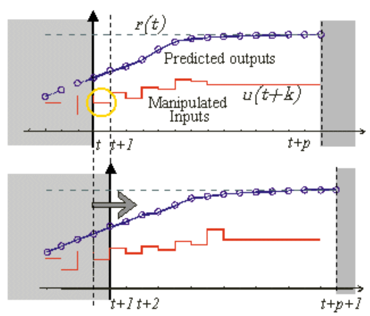


Image from http://www.dii.unisi.it/~bemporad/mpc.htm, A. Bemporad

Optimization through MPC

- 3D PWA system with state-space partition

$$x_{k+1} = A_j x_k + B_j u_k, \quad \text{if } x_k \in \Omega_j, \; j = 1,2,3,4$$

$$x_k \in \mathbb{X} = [-5,5]^3 \qquad u_k \in \mathbb{U} = [-2.5, 2.5]$$

$$Q = 0.02 I_3 \text{ and } R = 0.01$$

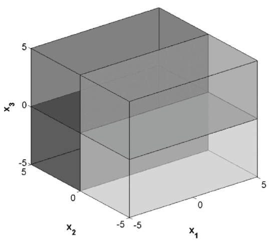
- Solution: Optimal control laws

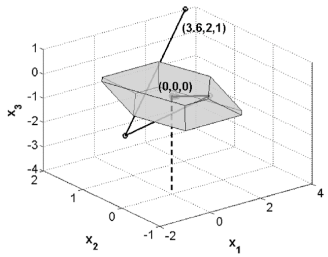
$$K_1 = [0.4699 \quad 0.1750 \quad 0.1591]$$
$$K_2 = [0.4039 \quad 0.4239 \quad 1.1529]$$
$$K_3 = [-0.7742 \quad -0.1436 \quad -0.1603]$$
$$K_4 = [-0.0800 \quad -0.0405 \quad -0.2867].$$

From Lazar, Heemels, Weiland, and Bemporad, TAC 2006

Optimization through MPC

- 3D PWA system with state-space partition (left)





- Computed optimal trajectory with computed terminal state set (right)

From Lazar, Heemels, Weiland, and Bemporad, TAC 2006

Summary

- Optimization of PWA systems with quadratic costs
  - Lower bounds through simple LMIs (Johansson and Rantzer)
  - Relaxed dynamic programming (Lincoln, Hedlund, and Rantzer)
  - Stabilizing MPC (Bemporad, Borelli, Morari, and others)
- Optimization of switching instants and sequences
  - Two-step process (Xu and Antsaklis)
  - Parameterization (Egerstedt, Wardi, and others)
- Very active area of research