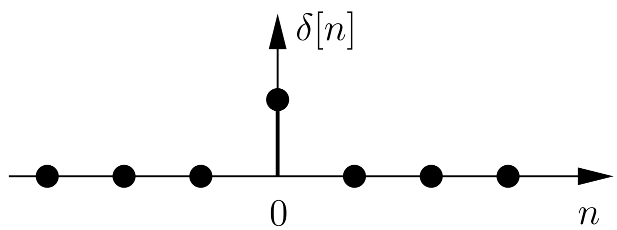
1.2.3 The Discrete–Time Unit Impulse and Unit Step Sequences

- *Unit impulse sequence* (or unit impulse or unit sample)

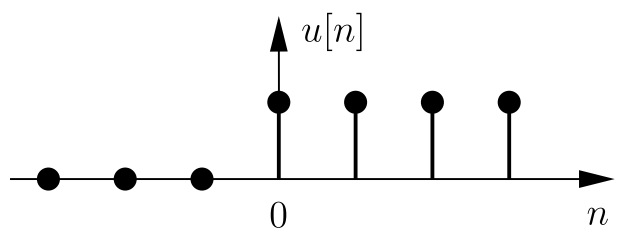
$$\delta[n] = \begin{cases} 1, & n = 0 \\ 0, & n \neq 0 \end{cases}$$

(also referred to as *Kronecker* delta function)



- *Unit step sequence* (unit step)

$$u[n] = \begin{cases} 1, & n \geq 0 \\ 0, & n < 0 \end{cases}$$



- Relation between $\delta[n]$ and $u[n]$
  - First order difference

$$\delta[n] = u[n] - u[n-1]$$

  - Running sum

$$u[n] = \sum_{m=-\infty}^{n} \delta[m]$$

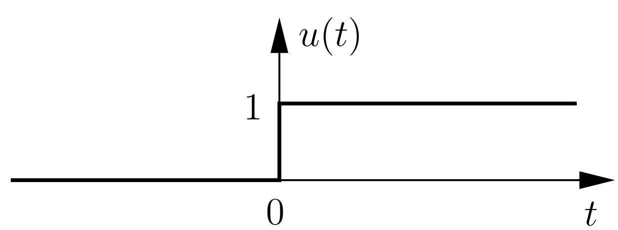
- *Sampling property* of unit impulse

$$x[n]\delta[n-n_0] = x[n_0]\delta[n-n_0]$$

1.2.4 The Continuous–Time Unit Impulse and Unit Step Functions

- *Unit step function* (unit step)

$$\boxed{u(t) = \begin{cases} 1, & t > 0 \\ 0, & t < 0 \end{cases}}$$



Note: discontinuity at $t = 0$

- *Unit impulse function* (unit impulse, Dirac delta impulse)

$$\delta(t) = \begin{cases} ?, & t = 0 \\ 0, & t \neq 0 \end{cases}$$

Remark:

We use the short-hand notation:

$$\frac{dx(t)}{dt} = \dot{x}(t)$$

- Relation between $\delta(t)$ and $u(t)$
  - First order derivative

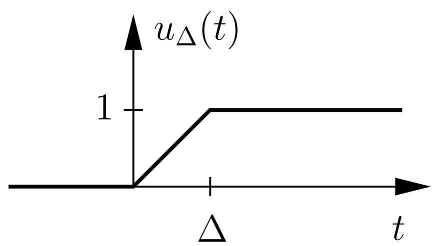
$$\delta(t) = \dot{u}(t)$$

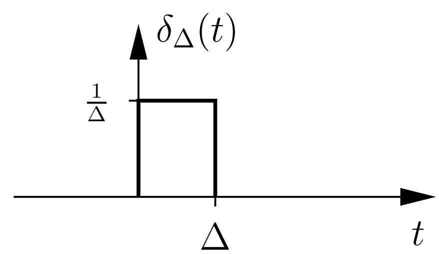
  - Running integral

$$u(t) = \int_{-\infty}^{t} \delta(\tau)\, d\tau$$

Formal difficulty: $u(t)$ is not differentiable in the conventional sense because of its discontinuity at $t = 0$.

- Some more thoughts on $\delta(t)$
  - Consider functions $u_\Delta(t)$ and $\delta_\Delta(t)$ instead of $u(t)$ and $\delta(t)$:



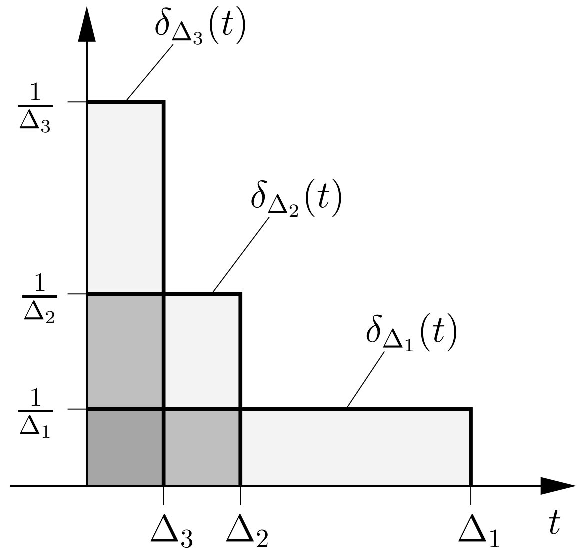


where

$$\delta_\Delta(t) = \dot{u}_\Delta(t)$$
$$u_\Delta(t) = \int_{-\infty}^{t} \delta_\Delta(\tau)\, d\tau$$

– Limit $\Delta \to 0$

* $u(t) = \lim\limits_{\Delta \to 0} u_\Delta(t)$
* $\delta(t)$:

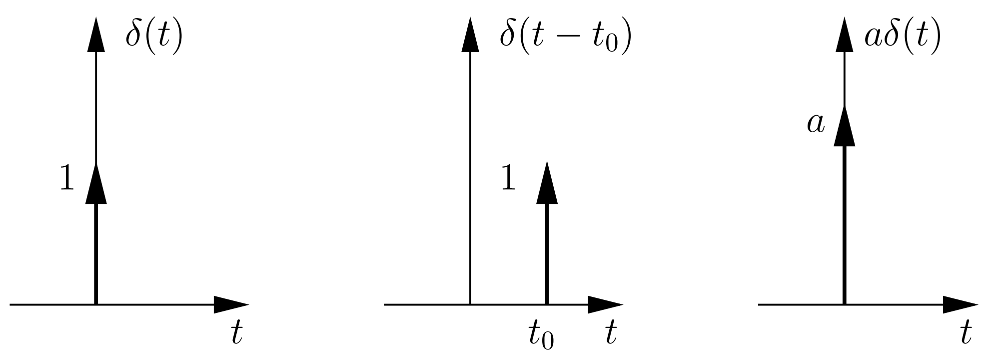


Observe: Area under $\delta_\Delta(t)$ always 1

$\Rightarrow$ $\delta(t)$ is an infinitesimally narrow impulse with area 1.

$$\boxed{\begin{aligned} \delta(t) &= \lim_{\Delta \to 0} \delta_\Delta(t) \\ \int_{-\infty}^{\infty} \delta(\tau)\, d\tau &= 1 \end{aligned}}$$

– Representation



– Properties

∗ Sampling property ($x(t)$ continuous at $t = t_0$)

$$\int_{-\infty}^{\infty} x(\tau)\delta(\tau - t_0)\, d\tau = x(t_0)$$

$$\boxed{x(t)\delta(t - t_0) = x(t_0)\delta(t - t_0)}$$

∗ Linearity

$$\begin{aligned}\int_{-\infty}^{\infty} (a\delta(\tau) + b\delta(\tau))x(\tau)\, d\tau &= \int_{-\infty}^{\infty} a\delta(\tau)x(\tau)\, d\tau + \int_{-\infty}^{\infty} b\delta(\tau)x(\tau)\, d\tau \\ &= (a + b)x(0)\end{aligned}$$

$$\boxed{a\delta(t) + b\delta(t) = (a + b)\delta(t)}$$

∗ Time scaling ($a \in \mathbb{R}$)

$$\int_{-\infty}^{\infty} \delta(a\tau)x(\tau)\, d\tau = \int_{-\infty}^{\infty} \frac{1}{|a|}\delta(\nu)x(\nu/a)\, d\nu = \frac{1}{|a|}x(0)$$

$$\boxed{\delta(at) = \frac{1}{|a|}\delta(t)}$$

$*$ Differentiation and derivative

$$\int_{-\infty}^{\infty} \dot{\delta}(\tau)x(\tau)\,d\tau = \delta(t)x(t)\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \delta(\tau)\dot{x}(\tau)\,d\tau = -\dot{x}(0)$$

$$\int_{-\infty}^{\infty} \dot{\delta}(\tau)x(\tau)\,d\tau = -\dot{x}(0)$$

$$\boxed{t\dot{\delta}(t) = -\delta(t)}$$

Remark:

More formal discussion of the unit impulse $\delta(t)$ in text books on *generalized functions* or *distributions*.